Forbidden Directed Minors and Directed Pathwidth

Shiva Kintali * Qiuyi Zhang †

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Abstract

Undirected graphs of pathwidth at most one are characterized by two forbidden minors i.e., (i) $K_3$, the complete graph on three vertices and (ii) $S_{2,2,2}$, the spider graph with three legs of length two each [BFKL87]. Directed pathwidth is a natural generalization of pathwidth to digraphs. In this paper, we prove that digraphs of directed pathwidth at most one are characterized by a finite number of forbidden directed minors. To achieve our goal, we present a new decomposition theorem for digraphs of directed pathwidth $\geq 2$ and prove several properties of the forbidden directed minors, which are of independent interest.

Keywords: directed pathwidth, forbidden minors, graph minors, pathwidth.

1 Introduction

Pathwidth measures how close an undirected graph is to being a path. Directed pathwidth, introduced by Reed, Seymour and Thomas (see [Bar06]), measures how close a digraph is to being a directed path. For an undirected graph (resp. digraph) $G$, let $\text{pw}(G)$ (resp. $\text{dpw}(G)$) denote its pathwidth (resp. directed pathwidth). Directed pathwidth is a generalization of pathwidth to digraphs i.e., for an undirected graph $G$, let $\overleftrightarrow{G}$ be the digraph obtained by replacing each edge $\{u, v\}$ of $G$ by two directed edges $(u, v)$ and $(v, u)$, then, $\text{dpw}(\overleftrightarrow{G}) = \text{pw}(G)$ [Bar06, Lemma 1].

Pathwidth has several equivalent characterizations in terms of vertex separation number, node searching number, interval thickness (i.e., one less than the maximum clique size in an interval supergraph) and cops and (invisible and eager) robber games. Similarly, directed pathwidth is characterized by directed vertex separation number and cops and (invisible and eager) robber games on digraphs [YC08]. Both pathwidth and directed pathwidth have associated decomposition structures called path decomposition and directed path decomposition respectively.

The graph minor theorem [RS04] (i.e., undirected graphs are well-quasi-ordered under the minor relation) implies that every minor-closed family of undirected graphs has a finite set of minimal forbidden minors. In particular, it implies that for all $k \geq 0$, graphs of treewidth (or pathwidth) $\leq k$ are characterized by a finite set of forbidden minors. The complete sets of forbidden minors are known for small values of treewidth and pathwidth. A graph has treewidth at most one (resp. two) if and only if it is $K_3$-free (resp. $K_4$-free). Graphs of treewidth at most three are characterized by four forbidden minors ($K_5$, the graph of the octahedron, the graph of the pentagonal prism,

*Department of Computer Science, Princeton University, Princeton, NJ 08540. Email: kintali@cs.princeton.edu
†Mathematics Department, Princeton University, Princeton, NJ 08540. Email: qiuyiz@princeton.edu
and the Wagner graph) [APC90, ST90]. Graphs of pathwidth at most one are characterized by two forbidden minors [BFKL87] i.e., (i) $K_3$ the complete graph on three vertices and (ii) $S_{2,2,2}$ the spider graph with three legs of length two each (see Figure 1). Graphs of pathwidth at most two are characterized by 110 forbidden minors [KL94].

A natural question is “are digraphs of bounded directed pathwidth characterized by a finite set of forbidden directed minors?” Unfortunately there is no generalization of the graph minor theorem for digraphs yet. Existing notions of directed minors (e.g. directed topological minors [Hun07], butterfly minors [JRST01], strong contractions [KS12], directed immersions [CS11]) do not imply well-quasi-ordering of all digraphs (see [Kin13] for more details).

Recently, the first author [Kin13] introduced a notion of directed minors based on several operations of contracting special subsets of directed edges, conjectured that digraphs are well-quasi-ordered under the proposed directed minor relation and proved the conjecture for some special classes of digraphs. This conjecture implies that every family of digraphs closed under the directed minor operations (see Section 2) are characterized by a finite set of minimal forbidden directed minors. In particular, it implies that for all $k \geq 0$, digraphs of Kelly-width (or DAG-width, or directed pathwidth) $\leq k$ are characterized by a finite set of forbidden directed minors (see [Kin13] for more details).

Using this directed minor relation, the authors studied the forbidden directed minors for digraphs with Kelly-width one and two (i.e., partial 0-DAGs and partial 1-DAGs) and proved that partial 0-DAGs (simply called DAGs) are characterized by one forbidden directed minor (i.e., $K_2$) and partial 1-DAGs are characterized by three forbidden directed minors (see [KZ13] for more details). Kelly-width is a generalization of treewidth.

A digraph has directed pathwidth zero if and only if it is a DAG and hence it is characterized by one minimal forbidden minor i.e., $K_2$ (see [KZ13]). In this paper, we show the finiteness of the set of minimal forbidden directed minors of digraphs with directed pathwidth one.

### 1.1 Notation and Preliminaries

We use standard graph theory notation and terminology (see [Die05]). For a directed graph (di-graph) $G$, we write $V(G)$ for its vertex set (or node set) and $E(G)$ for its edge set. All digraphs in this paper are finite and simple (i.e., no self loops and no multiple edges) unless otherwise stated. For $X \subseteq V(G)$, let $|X|$ denote the number of vertices in $X$. For an edge $e = (u, v)$, we say that $e$ is an edge from $u$ to $v$. We say that $u$ is the tail of $e$ and $v$ is the head of $e$. We also say that $u$ is an in-neighbor of $v$ and $v$ is an out-neighbor of $u$. The out-neighbors of a vertex $u$ is given by...
\[ N_{out}(u) = \{v : (u,v) \in E\} \] and the in-neighbors of a vertex \( v \) is given by \( N_{in}(v) = \{u : (u,v) \in E\}. \]

Let \( d_{out}(u) = |N_{out}(u)| \) (resp. \( d_{in}(v) = |N_{in}(v)| \)) denote the out-degree of \( u \) (resp. in-degree of \( v \)).

For \( S \subseteq V(G) \) we write \( G[S] \) for the subgraph induced by \( S \), and \( G \setminus S \) for the subgraph induced by \( V(G) \setminus S \). For \( F \subseteq E(G) \), we write \( G[F] \) for the subgraph with vertex set equal to the set of endpoints of \( F \), and edge set equal to \( F \).

For a digraph \( G \), let \( \overline{G} \) be the undirected graph, where \( V(\overline{G}) = V(G) \) and \( E(\overline{G}) = \{(u,v) : (u,v) \in E(G)\} \). We say that \( \overline{G} \) is the underline undirected graph of \( G \). For an undirected graph \( G \), let \( \hat{G} \) be the digraph obtained by replacing each edge \( \{u,v\} \) of \( G \) by two directed edges \( (u,v) \) and \( (v,u) \). We say that \( \hat{G} \) is the bidirected graph of \( G \).

A directed (simple) path in \( G \) is a sequence of vertices \( v_1, v_2, \ldots, v_l \) such that for all \( 1 \leq i \leq l - 1 \), \( (v_i, v_{i+1}) \in E(G) \). For a subset \( X \subseteq V(G) \), the set of vertices reachable from \( X \) is defined as:

\[
\text{Reach}_G(X) := \{v \in V(G) : \text{there is a directed path to } v \text{ from some } u \in X\}. \]

We say that \( G \) is weakly connected if \( \overline{G} \) is connected. We say that \( G \) is strongly connected if, for every pair of vertices \( u, v \in V(G) \), there is a directed path from \( u \) to \( v \) and a directed path from \( v \) to \( u \).

We use the term DAG when referring to directed acyclic graphs. Let \( T \) be a DAG. For two distinct nodes \( i \) and \( j \) of \( T \), we write \( i \prec_T j \) if there is a directed walk in \( T \) with first node \( i \) and last node \( j \). For convenience, we write \( i \preceq j \) whenever \( T \) is clear from the context. For nodes \( i \) and \( j \) of \( T \), we write \( i \preceq j \) if either \( i = j \) or \( i \prec j \). For an edge \( e = (i,j) \) and a node \( k \) of \( T \), we write \( e \prec k \) if either \( j = k \) or \( j \prec k \). We write \( e \sim i \) (resp. \( e \sim j \)) to mean that \( e \) is incident with \( i \) (resp. \( j \)).

Let \( W = (W_i)_{i \in V(T)} \) be a family of finite sets called node bags, which associates each node \( i \) of \( T \) to a node bag \( W_i \). We write \( W_{\geq i} \) to denote \( \bigcup_{j \geq i} W_j \).

The following notion of guarding is useful:

**Definition 1.** [Guarding] Let \( G \) be a digraph and \( W, X \subseteq V(G) \). We say \( X \) guards \( W \) if \( W \cap X = \emptyset \), and for all \( (u,v) \in E(G) \), if \( u \in W \) then \( v \in W \cup X \).

In other words, \( X \) guards \( W \) means that there is no directed path in \( G \setminus X \) that starts from \( W \) and leaves \( W \).

**Definition 2.** [Edge contraction] Let \( G \) be an undirected graph, and \( e = (u,v) \in E(G) \). The vertices and edges of the graph \( G' \) obtained from \( G \) by contracting \( e \) are:

- \( V(G') = V(G) \setminus u \)
- \( E(G') = (E(G) \cup \{(x,v) : (x,u) \in E(G)\}) \setminus \{(x,u) : x \in V(G)\} \)

**Definition 3.** [Minor] Let \( G \) and \( H \) be undirected graphs. We say that \( H \) is a minor of \( G \), (denoted by \( H \leq G \)), if \( H \) is isomorphic to a graph obtained from \( G \) by a sequence of vertex deletions, edge deletions and edge contractions. These operations may be applied to \( G \) in any order, to obtain its directed minor \( H \).

**Theorem 4.** (Robertson-Seymour theorem [RS04]) Undirected graphs are well-quasi-ordered by the minor relation \( \leq \).

**Corollary 5.** Every minor-closed family of undirected graphs has a finite set of minimal forbidden minors. In particular, for all \( k \geq 0 \), graphs of treewidth (or pathwidth) \( \leq k \) are characterized by a finite set of forbidden minors.
2 Directed Minors

In this section, we present a subset of the directed minor operations from [Kin13]. The directed minor relation in [Kin13] has more operations that are not necessary for our results in the current paper.

When we perform the following operations on a digraph $G$ to obtain a digraph $G'$, we remove any resulting self-loops and multi-edges from $G'$. Cycle contraction is a generalization of the edge contraction from Definition 2.

Definition 6. [Cycle contraction] Let $G$ be a graph, and $C = \{v_1, v_2, \ldots, v_l\} \subseteq V(G)$ be a directed cycle in $G$ i.e., $(v_i, v_{i+1}) \in E(G)$ for $1 \leq i \leq l - 1$ and $(v_l, v_1) \in E(G)$. The vertices and edges of the graph $G'$ obtained from $G$ by contracting $C$ are:

- $V(G') = V(G) \setminus C \cup \{w\}$, where $w$ is a new vertex.
- $E(G') = E(G) \setminus \{(x, v_i) : 1 \leq i \leq l, x \in V(G)\} \cup \{(v_i, x) : 1 \leq i \leq l, x \in V(G)\}$
  \[ \cup \{(x, w) : (x, v_i) \in E(G) \text{ for some } x \in V(G) \setminus C, 1 \leq i \leq l\} \cup \{(w, x) : (v_i, x) \in E(G) \text{ for some } x \in V(G) \setminus C, 1 \leq i \leq l\}\]

Butterfly contractions (defined by Johnson et al [JRST01]) allow us to contract a directed edge $e = (u, v)$ if either $e$ is the only edge with head $v$, or it is the only edge with tail $u$, or both. The following operations, out-contraction and in-contraction, are slightly general and allows any edge to be out-contracted or in-contracted after removing certain incident edges. Out-contracting an edge $(u, v)$ is equivalent to removing all the out-going edges of $u$ and identifying $u$ and $v$. Similarly, in-contracting an edge $(u, v)$ is equivalent to removing all the in-coming edges of $v$ and identifying $u$ and $v$. Hence, we can out-contract or in-contract any edge of $G$ without creating new paths in $G$.

Definition 7. [Out contraction] Let $G$ be a graph, and $e = (u, v) \in E(G)$. The vertices and edges of the graph $G'$ obtained from $G$ by out-contracting $e$ are:

- $V(G') = V(G) \setminus u$
- $E(G') = E(G) \cup \{(x, v) : (x, u) \in E(G)\} \setminus \{(u, x), (u, x) : x \in V(G)\}$

Note that we delete the vertex $u$, the tail of the edge $e = (u, v)$. The vertex $v$ exists in both $G$ and $G'$.

Definition 8. [In contraction] Let $G$ be a graph, and $e = (u, v) \in E(G)$. The vertices and edges of the graph $G'$ obtained from $G$ by in-contracting $e$ are:

- $V(G') = V(G) \setminus v$
- $E(G') = E(G) \cup \{(u, x) : (v, x) \in E(G)\} \setminus \{(x, v), (v, x) : x \in V(G)\}$

Note that we delete the vertex $v$, the head of the edge $e = (u, v)$. The vertex $u$ exists in both $G$ and $G'$.

Definition 9. [Directed minor] Let $G$ and $H$ be digraphs. We say that $H$ is a directed minor of $G$, (denoted by $H \preceq G$), if $H$ is isomorphic to a graph obtained from $G$ by a sequence of vertex deletions, edge deletions, cycle contractions, out/in contractions. These operations may be applied to $G$ in any order, to obtain its directed minor $H$. 

4
3 Directed Pathwidth

3.1 Definition

Definition 10. A Directed path decomposition of a digraph $G$ is a sequence $X_1, X_2, \ldots, X_r$ of subsets (node bags) of $V(G)$, such that:

- $\bigcup_{1 \leq i \leq r} X_i = V(G)$. (DPW-1)
- For all $i, j, k \in [r]$, if $i \leq j \leq k$, then $X_i \cap X_k \subseteq X_j$. (DPW-2)
- For all arcs $(u, v) \in E(G)$, there exist $i, j$ with $1 \leq i \leq j \leq r$ such that $u \in X_i$ and $v \in X_j$. (DPW-3)

The width of a directed path decomposition $X = (X_i)_{i \in [r]}$ is defined as $\max\{|X_i| : 1 \leq i \leq r\} - 1$. The directed pathwidth of $G$, denoted by $\text{dpw}(G)$, is the minimum width over all possible directed path decompositions of $G$.

(DPW-2) can be replaced by the following equivalent statement:

- For any $v \in V(G)$, $\{i : X_i \cap v \neq \emptyset, 1 \leq i \leq r\}$ is an integer interval. (DPW-2′)

(DPW-3) can be replaced by the following equivalent statement:

- For any $i$ with $1 < i < r$, there is no edge from $\bigcup_{i+1 \leq j \leq r} X_j$ to $\bigcup_{1 \leq j \leq i-1} X_j$ in $G \setminus X_i$. (DPW-3′)

A directed path decomposition is called incremental if it satisfies the following condition:

- For any $i \in [r-1]$, $|X_i \Delta X_{i+1}| = 1$. (DPW-4)

3.2 Basic properties

A digraph $D$ is a directed union of digraphs $D_1$ and $D_2$ if $D_1$ and $D_2$ are induced subgraphs of $D$, $V(D_1) \cup V(D_2) = V(D)$, and no edge of $D$ has head in $V(D_1)$ and tail in $V(D_2)$. Directed pathwidth is closed under directed unions (see [YC08]). The following theorem is immediate.

Theorem 11. ([YC08]) Directed pathwidth of a digraph $G$ is equal to the maximum directed pathwidth taken over the strongly-connected components of $G$.

Hence, all digraphs in the rest of this paper are strongly connected, unless otherwise stated. Kim and Seymour [KS12, Kim13] proved that directed pathwidth is closed under strong minors and butterfly minors. The following lemma is immediate.

Lemma 12. Let $H \preceq G$. Then, $\text{dpw}(H) \leq \text{dpw}(G)$. In other words, directed pathwidth is monotone under the directed minor operations mentioned in Section 2.
4 Basic Forbidden Minors

Let \( F \) be the set of all minor-minimal (w.r.t \( \leq \)) digraphs of directed pathwidth two. Our goal is to prove that \( F \) is finite. By Theorem 11 we may assume that all digraphs in \( F \) are strongly-connected.

For a digraph \( G \), let \( \bar{G} \) be the digraph obtained by reversing the directions of all the edges in \( G \).

**Lemma 13.** ([YC08, Tam11]) For any digraph \( G \), \( \text{pw}(G) = \text{pw}(\bar{G}) \).

**Lemma 14.** For digraphs \( G \) and \( H \), \( H \preceq G \) if and only if \( \bar{H} \preceq \bar{G} \).

**Theorem 15.** (Kintali, Zhang [KZ13]) Let \( G \) be a simple digraph such that every vertex in \( V(G) \) has out-degree \( \geq 2 \). Then \( G \) contains \( K_3, \bar{N}_4 \) or \( \bar{M}_5 \) (see Figure 2) as a directed minor.

**Theorem 16.** Let \( G \) be a simple digraph such that every vertex in \( V(G) \) has in-degree \( \geq 2 \). Then \( G \) contains \( K_3, \bar{N}_4 \) or \( \bar{M}_5 \) as a directed minor.

**Corollary 17.** \( K_3, N_4, M_5, \bar{N}_4, \bar{M}_5 \in F \). We say that these five digraphs are the “basic forbidden minors”.

![Figure 2: K₃, N₄ and M₅](image)

Let \( F' = F \setminus \{K_3, N_4, M_5, \bar{N}_4, \bar{M}_5\} \). Since \( \text{dpw}(S_{2,2,2}) = 2 \), (and \( S_{2,2,2} \) is a forbidden minor for undirected graphs of pathwidth at most one) we have \( S_{2,2,2} \in F' \). So \( F' \) is non-empty. The rest of this paper is aimed at proving that \( F' \) is finite. Let \( G \in F' \). The following three properties are crucial.

- \( G \) is strongly-connected (PROP-1)
- \( G \) has at least one vertex of out-degree equal to one (PROP-2)
- \( G \) has at least one vertex of in-degree equal to one (PROP-3)

5 A New Decomposition Theorem

In this section, we present a new structural characterization of digraphs with directed pathwidth \( \geq 2 \). For \( X \subseteq V(G) \), the complement of \( X \) is defined as \( X^c := V(G) \setminus X \). For \( X, Y \subseteq V(G) \) we say that \( X \) hits \( Y \) if \( Y \setminus X \) guards \( X \). \( X \) has an out-neighbor in \( Y \) if there exists \( u \in X \) and \( v \in Y \) such that \( (u, v) \in E(G) \). A vertex \( u \) has \( k \) out-neighbors in \( Y \) if there exist distinct \( v_1, ..., v_k \in Y \) such that \( (u, v_i) \in E(G) \) for all \( i \in [k] \).
### 5.1 Elimination Orderings

**Definition 18.** Let $G$ be a digraph and $u \in V(G)$ with $d_{\text{out}}(u) = 1$, then $R(G, u)$ is defined as the digraph obtained by out-contracting $u$. We say that $R(G, u)$ is obtained by eliminating $u$ or reducing $u$ in $G$. If $d_{\text{out}}(u) \neq 1$, then $R(G, u)$ is not defined.

In other words, eliminating or reducing a vertex $u$ in a digraph $G$ yields digraph $G' := R(G, u)$ that is obtained from $G$ by deleting vertex $u$ and adding arcs from all in-neighbors of $u$ to the unique out-neighbor of $u$, without introducing loops or multiple edges. In the rest of this paper whenever we use the notation $R(G, u)$, we assume that $d_{\text{out}}(u) = 1$ in $G$.

**Observation 19.** If $G$ is strongly-connected then $R(G, u)$ is also strongly-connected.

**Definition 20.** We say that $u_1, \ldots, u_m$ is an elimination ordering in $G$, if $G_0 := G$ and $G_i := R(G_{i-1}, u_i)$ is defined for $i \in [m]$ i.e., $d_{\text{out}}(u_i) = 1$ in $G_{i-1}$. The corresponding out-sequence for this elimination ordering is a sequence of vertices $v_1, \ldots, v_m$, where $v_i$ is the unique out-neighbor of $u_i$ in $G_{i-1}$ for $i \in [m]$. We also denote $G_i$ as $R(G, [u_1, \ldots, u_i])$.

**Observation 21.** Let $u_1, \ldots, u_m$ be an elimination ordering in $G$ and $v_1, \ldots, v_m$ be its out-sequence, then $N_{\text{out}}^G(u_i) \subseteq \{u_1, \ldots, u_{i-1}, v_1, \ldots, v_{i-1}, v_i\}$.

**Definition 22.** Let $u_1, \ldots, u_m$ be an elimination ordering in $G$ and $v_1, \ldots, v_m$ be its out-sequence. We say that this elimination ordering is linear if either $u_{i+1} = v_i$ or $v_{i+1} = v_i$ holds for $1 \leq i < m$.

**Observation 23.** Let $u_1, \ldots, u_m$ be a linear elimination ordering and $v_1, \ldots, v_m$ be its corresponding out-sequence, then $v_i \in \{u_1, \ldots, u_m, v_m\}$ for $i \in [m]$.

**Lemma 24.** Let $u_1, \ldots, u_m$ be a linear elimination ordering and $v_1, \ldots, v_m$ be its corresponding out-sequence, then $\{u_1, \ldots, u_i\}$ hits $\{v_i\}$ for all $1 \leq i \leq m$.

**Proof.** We proceed with induction on $i$ for $1 \leq i \leq m$. Note that $\{u_1\}$ hits $\{v_1\}$ since $u_1$ has only one out-neighbor. Assume that $\{u_1, \ldots, u_{i-1}\}$ hits $\{v_{i-1}\}$. Since $u_i$ has out-degree 1 at time of contraction, we have $N_{\text{out}}(u_i) \subseteq \{u_1, \ldots, u_{i-1}, v_i\}$. By linearity, either $v_{i-1} = u_i$ or $v_{i-1} = v_i$.

**Case 1:** $v_{i-1} = u_i$. Now, consider any $w \in \{u_1, \ldots, u_i\}$. If $w \neq u_i$, then by the inductive hypothesis, $N_{\text{out}}(w) \subseteq \{u_1, \ldots, u_{i-1}, u_i\}$. If $w = u_i$, then $N_{\text{out}}(w) \subseteq \{u_1, \ldots, u_{i-1}, v_i\}$. We conclude that $\{u_1, \ldots, u_i\}$ hits $\{v_i\}$.

**Case 2:** $v_{i-1} = v_i$. Now, consider any $w \in \{u_1, \ldots, u_i\}$. If $w \neq u_i$, then by the inductive hypothesis, $N_{\text{out}}(w) \subseteq \{u_1, \ldots, u_{i-1}, v_i\}$. If $w = u_i$, then $N_{\text{out}}(w) \subseteq \{u_1, \ldots, u_{i-1}, v_i\}$. We conclude that $\{u_1, \ldots, u_i\}$ hits $\{v_i\}$. \(\square\)

In the rest of this paper, all elimination orderings are linear. We say a digraph $G$ admits an elimination ordering if there exists an elimination ordering $u_1, \ldots, u_m$ such that $R(G, [u_1, \ldots, u_m])$ is a single vertex.

**Lemma 25.** A strongly connected digraph $G$ has directed pathwidth at most 1 if and only if it admits an elimination ordering. In other words, digraphs in $\mathcal{F}$ do not admit an elimination ordering.

**Definition 26.** An elimination ordering $u_1, \ldots, u_m$ in $G$ is maximal if there does not exist $u_{m+1} \in V(G)$ such that $u_1, \ldots, u_{m+1}$ is also an elimination ordering in $G$. 

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Observation 27. Let $G \in \mathcal{F}$. Let $u_1, \ldots, u_m$ be an elimination ordering in $G$ and $v_1, \ldots, v_m$ be its corresponding out-sequence. It is maximal if and only if in $R(G, [u_1, \ldots, u_m])$, $v_m$ has outdegree $\geq 2$ and all in-neighbors of $v_m$ have outdegree $\geq 2$.

Lemma 28. Let $u_1, \ldots, u_m$ be a maximal elimination ordering and let $u'_1, \ldots, u'_l$ be an elimination ordering. If $u'_i \in \{u_1, \ldots, u_m\}$, then $u'_i \in \{u_1, \ldots, u_m\}$ for all $1 \leq i \leq l$.

Proof. Let $u'_1, \ldots, u'_l$ be an elimination ordering and $v'_1, \ldots, v'_l$ be its corresponding out-sequence. We use strong induction on $i$. For the base case of $i = 1$, it holds by assumption. Now, assume that $u'_j \in \{u_1, \ldots, u_m\}$ for $1 \leq j < i$.

For the sake of contradiction, assume $u'_i \notin \{u_1, \ldots, u_m\}$. Since $u'_i$ can be contracted in $R(G, [u'_1, \ldots, u'_{i-1}])$, we see that in $G$, $N_{out}(u'_i) \subseteq \{u'_1, \ldots, u'_{i-1}, v'_i\}$. Also, by Lemma 24, we see that $\{u'_1, \ldots, u'_{i-1}\}$ hits $\{v'_i\}$. Thus, since $u'_1, \ldots, u'_{i-1}$ were reduced in the elimination ordering $u_1, \ldots, u_m$, we see that in $R(G, [u_1, \ldots, u_m])$, the set $\{u'_1, \ldots, u'_{i-1}, v'_i\}$ must have been contracted to at most 1 vertex. Now, by linearity, either $v'_i = v'_{i-1}$ or $u'_i = v'_{i-1}$.

Case 1: $v'_i = v'_{i-1}$. Then, $N_{out}(u'_i) \subseteq \{u'_1, \ldots, u'_{i-1}, v'_i\}$. In $R(G, [u_1, \ldots, u_m])$, since $\{u'_1, \ldots, u'_{i-1}, v'_i\}$ is now at most 1 vertex, $d_{out}(u'_i) = 1$ and in fact, $N_{out}(u'_i) = \{v_m\}$. Therefore, we may let $u_{m+1} = u'_i$ and $v_{m+1} = v'_i$, contradicting maximality.

Case 2: $u'_i = v'_{i-1}$. Then, in $R(G, [u_1, \ldots, u_m])$, $\{u'_1, \ldots, u'_{i-1}, u'_i\}$ is now at most 1 vertex, and the vertex is $u'_i$. In fact, we can find $k$ such that $v_k = u'_i$ and by linearity, we must have $u'_k = v_m$. Since $N_{out}(u'_i) \subseteq \{u'_1, \ldots, u'_{i-1}, v'_i\}$, we see that $d_{out}(u'_i) = 1$ and letting $u_{m+1} = u'_i = v_m$ contradicts maximality. \qed

Lemma 29. Let $u_1, \ldots, u_m$ be a maximal elimination ordering and $u'_1, \ldots, u'_l$ be another maximal elimination ordering with $u_1 = u'_1$. Then, $l = m$ and $u'_1, \ldots, u'_l$ is a permutation of $u_1, \ldots, u_m$. Furthermore, if $v_1, \ldots, v_m$ and $v_1', \ldots, v'_l$ are the corresponding out-sequences respectively, then $v_m = v'_l$.

Proof. Since $u_1 \in \{u'_1, \ldots, u'_l\}$ and $u'_1 \in \{u_1, \ldots, u_m\}$, by Lemma 28, we see that $\{u_1, \ldots, u_m\}$ and $\{u'_1, \ldots, u'_l\}$ must contain each other and thus have the same elements. Hence $u'_1, \ldots, u'_l$ is a permutation of $u_1, \ldots, u_m$ and $l = m$.

Let $X = \{u_1, \ldots, u_m\} = \{u'_1, \ldots, u'_l\}$. By Lemma 24, $X$ hits $v_m$ and $X$ hits $v'_l$. Also, $v_m, v'_l \notin X$. Hence, $v_m = v'_l$. \qed

Lemma 30. If $u_1, \ldots, u_m$ and $w_1, \ldots, w_m$ are two elimination orderings that are permutations of each other, then $R(G, [u_1, \ldots, u_m]) = R(G, [w_1, \ldots, w_m])$.

5.2 Elimination Sets

Definition 31. Let $u_1, \ldots, u_m$ be the unique maximal (up to permutation) elimination ordering starting at $u_1 = u$ and let $v_1, \ldots, v_m$ be its out-sequence. We say that $S_u = \bigcup_{i=1}^m \{u_i, v_i\}$ is the elimination set of $u$. The unique “end vertex” $v_m$ is called the bad vertex of $S_u$. We denote this bad vertex by $b_u := v_m$. Note that $S_u = \{v_m\} \cup \bigcup_{i=1}^m \{u_i\}$.

Definition 32. Let $S_u$ be the elimination set of $u$, then $R(G, S_u)$ is defined to be the unique digraph (uniqueness is implied by Lemma 30) obtained by eliminating vertices according to a maximal elimination ordering starting at $u$.

In the rest of this paper, all elimination orderings are maximal.
Lemma 33. If \( x \in S_u, y \in S_u^c \) and \( (x, y) \in E(G) \), then \( x = b_u \).

Proof. Lemma 24 implies that \( S_u \setminus \{b_u\} \) hits \( \{b_u\} \). The lemma follows.

Lemma 34. Let \( S_u \) be the elimination set of \( u \). Then, \( b_u \) has at least 2 out-neighbors in \( S_u^c \). Furthermore, if \( v \in S_u^c \) has out-neighbors in \( S_u \), then \( v \) has out-neighbors in \( S_u^c \).

Proof. By the maximality of the elimination ordering, \( b_u \) must have at least 2 out-neighbors in \( S_u^c \). If \( v \in S_u^c \) has out-neighbors in \( S_u \), then \( v \) is an in-neighbor of \( b_u \) in \( R(G, S_u) \). Again by the maximality of the elimination ordering \( v \) must have out-neighbors in \( S_u^c \).

Lemma 35. Let \( u \) be a vertex of out-degree 1 and \( v \) be its unique out-neighbor. Let \( w \) be another vertex. If \( S_w \) contains \( u \) or \( v \), then \( S_u \subseteq S_w \).

Proof. By maximality, we see that if \( S_w \) contains either \( u \) or \( v \), then \( u \) must have been eliminated in the elimination ordering starting at \( w \). Let \( u_1, ..., u_m \) be the elimination ordering starting at \( u_1 = u \). Therefore, by Lemma 28, we see that \( \{u_1, ..., u_m\} \subseteq S_w \). By Lemma 24, \( \{u_1, ..., u_m\} \) hits \( \{v_m\} \), so we see that \( v_m \in S_w \). We conclude that \( S_u \subseteq S_w \).

5.3 Elimination Decomposition

Let \( G \) be a strongly-connected digraph. Let \( |V(G)| = n \) and \( V(G) = \{w_1, ..., w_n\} \). Let \( \{S_{w_1}, ..., S_{w_n}\} \) be the set of all elimination sets in \( G \). Each elimination set \( S_{w_i} \) is a subset of \( V(G) \). If there are sets \( S_{w_i} = S_{w_j} \), then we remove one of them arbitrarily. Let \( S = \{S_{v_1}, ..., S_{v_l}\} \) be the set of elimination sets without duplicates. The subset relation defines a natural partial order on the sets in \( S \). Let \( S_* = \{S_{u_1}, ..., S_{u_k}\} \) be the largest anti-chain of the sets in \( S \).

Definition 36. We say that \( S_* = \{S_{u_1}, ..., S_{u_k}\} \) is an elimination decomposition of \( G \). We define \( S_0 := V(G) \setminus \bigcup_{i=1}^k S_{u_i} \)

As an example, we now exhibit an elimination decomposition of \( G = S_{2,2,2} \), the spider graph with three legs of length two each. \( S_{2,2,2} \) has three legs connected by a vertex (say \( w \)). Let \( u_i, v_i \) be the vertices in the \( i \)-th branch such that \( (u_i, v_i), (v_i, u_i), (v_i, w), (w, v_i) \in E(G) \). Then, the elimination sets are \( S_{u_i} = \{u_i, v_i, w\} \) for \( 1 \leq i \leq 3 \). \( S_* = \{S_{u_1}, S_{u_2}, S_{u_3}\} \) is the elimination decomposition of \( G \).

Theorem 37 (Characterization Theorem). Let \( G \) be a strongly connected digraph. Then, \( G \) has directed pathwidth \( \geq 2 \) if and only if \( V(G) \) can be decomposed into sets \( V_0, V_1, ..., V_s \), with \( V(G) = \bigcup_{i=0}^s V_i \) and satisfying the following properties:

- **Property 1:** for \( 1 \leq i \leq s \), \( V_i \) is a proper subset of \( V(G) \)
- **Property 2:** \( V_0 \) is a set of vertices of out-degree \( \geq 2 \)
- **Property 3:** for \( 1 \leq i \leq s \), if \( v_i \in V_i \) has out-neighbors in \( V_i^c \), then \( v_i \) has at least 2 such out-neighbors
- **Property 4:** for \( 1 \leq i \leq s \), if \( u \in V_i^c \) has out-neighbors in \( V_i \), then \( u \) has out-neighbors in \( V_i^c \).
Proof. If $G$ is of directed pathwidth $\geq 2$, then consider its elimination decomposition $S_{u_1}, ..., S_{u_s}$ and let $V_i = S_{u_i}$ and $V_0 = S_0$. By definition of $S_0$, we see that $\bigcup_{i=1}^{s} V_i = V(G)$. Note that if there exists $i$ such that $S_v = V(G)$, then $G$ would have directed pathwidth $\leq 1$, so $S_v$ must be a strict subset. If there exists $v \in S_0$ that has outdegree $\leq 1$, then $S_v$ is an elimination set, but it is not necessarily maximal or in the elimination decomposition. However, we see that $S_v \subseteq S_w$ for some $S_w$ in the elimination decomposition. But $v \in S_w$ contradicts $v \in S_0$. Properties 3 and 4 follow from Lemma 33 and Lemma 34.

If we can find such vertex sets $V_0, ... V_m$, then assume for contradiction that $G$ is of pathwidth $\leq 1$. Thus, since $G$ is strongly connected, we can find a linear elimination ordering $u_1, ..., u_m$ and its outsequence $v_1, ..., v_m$ such that $G$ is reduced to a single vertex. Since $u_1$ has outdegree 1, we see that $u_1 \in V_i$ for some $i \geq 1$. Since $V_i$ is only a strict subset of $V(G)$, there must exist $l$ minimal such that either $u_l \not\in V_i$ or $v_l \not\in V_i$.

Case 1: $u_l \not\in V_i$. Since $u_l$ can be contracted and the minimality of $l$, we know that $N_{out}(u_l) \subseteq \{u_1, ..., u_{l-1}, v_{l-1}\} \subseteq V_i$. Note that $u_l$ must have at least one out-neighbor in $V_i$, so this contradicts property 4.

Case 2: $v_l \not\in V_i$. Since $u_l \in V_i$ and $v_l \not\in V_i$, by property 3, $u_l$ has at least 2 out-neighbors in $V_i$. Since $\{u_1, ..., u_l\} \subseteq V_i$, we see that the two out-neighbors of $u_l$ in $V_i$ were not eliminated and $u_l$ has outdegree $\geq 2$ at time of elimination, a contradiction. \hfill $\square$

6 Intersection Properties

6.1 Lonely Sets

Definition 38. Let $S_* = \{S_{u_1}, ..., S_{u_s}\}$ be an elimination decomposition of $G$. Then, $X \subseteq V(G)$ is lonely if there exists $i$ such that $\bigcup_{j \neq i} S_{u_j} \cap X = \emptyset$.

Lemma 39. Let $S_u$ be an elimination set in $S_*$ and let $v$ be the unique out-neighbor of $u$. Then, \{u, v\} is lonely.

6.2 Double Intersection

Lemma 40. Let $S_{v_1}, ..., S_{v_t}$ be a subset of elimination sets in $S_*$ such that $S_{v_i}$ hits $S_{v_{i+1}}$ for $1 \leq i \leq t - 1$, and $S_{v_t}$ hits $S_{v_1}$. Then, $V(G) = \bigcup_{i=1}^{t} S_{v_i}$ and there are no other elimination sets in $S_*$.

Proof. Let $H = \bigcup_{i=1}^{t} S_{v_i}$. For any vertex $v \in S_{v_i}$, all its out-neighbors are in $S_{v_i} \cup S_{v_{i+1}}$. Hence, $H$ has no out-neighbors in $H^c$. By strong connectivity of $G$, $H^c = \emptyset$. \hfill $\square$

Lemma 41. If $S_u \cap S_v \neq \emptyset$, then $\{b_u, b_v\} \cap S_u \cap S_v \neq \emptyset$.

Proof. By strong connectivity, $S_u \cap S_v$ must have out-neighbors in $(S_u \cap S_v)^c = S_u^c \cup S_v^c$. Without loss of generality, assume it has out-neighbors in $S_u^c$, but by Lemma 33, the only vertex in $S_u$ that has out-neighbors in $S_u^c$ is $b_u$. Therefore, $b_u \in S_u \cap S_v$. \hfill $\square$

Corollary 42. For an elimination set $S_u$, if $\{b_u\}$ is lonely, then $S_u$ is lonely.

Lemma 43. If $b_u \in S_v$ and $b_u \not= b_v$, then $S_u$ hits $S_v$. 

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Lemma 33, we conclude that $u \in S_v$. Let $b_v \notin S_u$, we must have reduced $b_v$ during the elimination process starting at $v$. Hence, $N_{out}(b_u) \subseteq S_v$. By Lemma 33, $S_u$ hits $S_v$. 

**Corollary 44.** If $S_u \cap S_v \neq \emptyset$ and $b_v \notin S_u$, then $S_u$ hits $S_v$.

**Corollary 45.** If $b_u, b_v \in S_u \cap S_v$ and $b_u \neq b_v$, then $S_u$ and $S_v$ hit each other and $V(G) = S_u \cup S_v$.

### 6.3 Triple Intersection

Now we consider the case when there are three distinct elimination sets $S_u, S_v, S_w$ in $S$ that pairwise intersect. We say that $S_u \cap S_v \cap S_w$ is the “triple intersection”.

**Lemma 46.** If $S_u, S_v, S_w$ pairwise intersect and $S_u \cap S_v \cap S_w = \emptyset$, then $S_u, S_v$ and $S_w$ hit each other in a cycle and $V(G) = S_u \cup S_v \cup S_w$.

**Proof.** Consider $S_u, S_v$ and note that we cannot have $b_u, b_v \in S_u \cap S_v$ or else due to the empty triple intersection, we have $b_u, b_v \notin S_w$. But since $S_w$ intersects both $S_u, S_v$, we must have $b_w \in S_u, S_v$. But, $b_w$ is in the triple intersection, a contradiction. Since one of $b_u, b_v$ must be in the intersection, so without loss of generality, let $b_u \in S_v$ and $b_v \notin S_u$. By Corollary 44, $S_u$ hits $S_v$.

Similarly, $S_w$ intersects $S_u$, but since $b_u \in S_u \cap S_v$ and the triple intersection is empty, we conclude that $b_u \notin S_w$. By Corollary 44, we conclude that $S_w$ hits $S_u$ and by Lemma 41, $b_w \in S_u$. Lastly note that $b_w \notin S_v$ since the triple intersection is empty. We conclude that $S_v$ hits $S_w$. Thus, we have 3 elimination sets that hit each other and by Lemma 40, $V(G) = S_u \cup S_v \cup S_w$.

**Lemma 47.** If $S_u \cap S_v \cap S_w \neq \emptyset$, then one of $b_u, b_v, b_w$ must be in the triple intersection.

**Proof.** Let $H = S_u \cap S_v \cap S_w$. By strong connectivity of $G$, $H$ must have an out-neighbor in $H^c$. Let $u \in H, v \in H^c$ such that $(u, v) \in E(G)$. Note that either $v \in S_u^c$ or $v \in S_v^c$ or $v \in S_w^c$. By Lemma 33, we conclude that $u$ must be either $b_u$ or $b_v$ or $b_w$ respectively.

**Lemma 48.** If we have $S_u, S_v, S_w$ such that $b_u, b_v, b_w \in S_u \cap S_v \cap S_w$, then $b_u = b_v = b_w$.

**Proof.** By Corollary 45, if any of the bad vertices are not equal, then we have only two elimination sets, contradicting the existence of $S_u, S_v$ and $S_w$.

**Lemma 49.** If we have $S_u, S_v, S_w$ such that $S_u \cap S_v \cap S_w \neq \emptyset$ and $b_u$ is in the triple intersection and $b_v$ is not in the triple intersection, then $S_u$ hits $S_v$ and $S_w$.

1) If $b_u = b_w$, then $S_w$ hits $S_v$.

2) If $b_w = b_v$, then $S_u$ hits $S_w$.

3) If all vertices are distinct, then $S_u$ hits $S_w$. Also, $S_w$ hits $S_v$ or $S_v$ hits $S_w$.

**Proof.** Assume $b_u$ is in the triple intersection set $H = S_u \cap S_v \cap S_w$ and $b_v$ is not. Note first that $b_v \notin S_u$, or else by Lemma 43, $b_u \neq b_v$ implies we only have two elimination sets, contradicting existence of $S_w$. By Corollary 44, $S_u$ hits $S_v$.

If $b_u = b_w$, then $b_w \in H$ and $b_v \neq b_w$. As previous, we conclude that $S_w$ hits $S_v$.

If $b_w = b_v$, then we note that $b_w \neq b_u$, so $S_u$ hits $S_w$.

Otherwise, we see that $b_u, b_v, b_w$ are all distinct vertices and note also by Lemma 43, $b_v, b_w \notin S_u$. $b_w \neq b_u$ still gives $S_u$ hits $S_w$. Then, one of $b_v, b_w$ must be in $S_v \cap S_w$, so either $S_v$ hits $S_w$ or vice versa.
Lemma 50. If we have $S_u, S_v, S_w$ such that $S_u \cap S_v \cap S_w \neq \emptyset$ and $S_u$ hits $S_v$, then one of the following holds:

1) $b_u$ is in the triple intersection and $b_v$ is not
2) $b_w$ is in the triple intersection and $b_u, b_v$ are not and $S_w$ hits $S_u, S_v$.

Proof. Note that if $b_u = b_v$, then since $S_u$ hits $S_v$, all the out-neighbors of $b_u = b_v$ are contained in $S_u \cup S_v$, which means $S_u, S_v$ hit each other and by Lemma 40, this contradicts existence of $S_w$.

Also, if $b_v \in S_u$, then $b_u \neq b_v$ implies $S_v$ hits $S_u$, which again contradicts existence of $S_w$.

Therefore, $b_v \notin S_u$ and so $b_v$ is not in the triple intersection. If $b_u$ is in the triple intersection, we are done. Otherwise, $b_u, b_v$ are both not in the triple intersection and by Lemma 47, $b_w$ is in the triple intersection. By Lemma 49, $S_w$ hits both $S_u, S_v$. □

6.4 More Triple Intersections

In this section, we study intersections of the form $S_u \cap S_v \cap S_w^c$, under the assumption that the triple intersection is not empty.

Lemma 51. Let $S_u \cap S_v \neq \emptyset$ and let $u_1, \ldots, u_m$ be the elimination ordering of $S_u$ and $v_1, \ldots, v_m$ be the corresponding out-sequence. If $u_t = b_v$ and there exists $t \geq 1$ such that $v_t \in S_v$, then $S_u, S_v$ hit each other and $V(G) = S_u \cup S_v$.

Proof. We see that by Lemma 24, $\{u_1, \ldots, u_t\}$ hits $v_t$ in $G$. Since $u_t = b_v$ and $v_t \in S_v$, we see that by Lemma 33, $\{u_1, \ldots, u_t\} \cup S_v$ has no out-neighbors to its complement. So, by strong connectivity, $G = \{u_1, \ldots, u_t\} \cup S_v$. Therefore, $V(G) = S_u \cup S_v$ and there are no other vertices. Since there are only two elimination sets, $S_u, S_v$ hit each other trivially. □

Lemma 52. Let $S_u \cap S_v \neq \emptyset$ and let $u_1, \ldots, u_m$ be the elimination ordering of $S_u$ and $v_1, \ldots, v_m$ be the corresponding out-sequence. Assume there exists $t$ minimal such that $u_t \in S_u$ and there exists minimal $p \geq t$ such that $v_p \notin S_v$. If $(S_u \cup S_v)^c \neq \emptyset$, then $u_p = b_v$ and $u_b \notin S_v$.

Proof. We see that by definition, $v_t \in S_v$ for $t \leq l < p$. Now, by linearity, we see that either $v_{p-1} = v_p$ or $v_{p-1} = u_p$. Since $v_{p-1} \in S_v$, $v_{p-1} = u_p$ must hold. Realize $b_v$ cannot be eliminated before $v_p$, since if $p = t$, then minimality of $t$ is contradicted and if $p > t$, since $v_{p-1} \in S_v$, by Lemma 51, we see that $(S_u \cup S_v)^c = \emptyset$, a contradiction.

Since $u_p \in S_v$ and $v_p \notin S_v$ and $b_v$ was not eliminated, we conclude that $u_p = b_v$. Also, we claim $b_u \notin S_v$. If $u_b \in S_u$ and since $b_u \in S_u$, then if $u_b \neq b_v$, we have $S_u, S_v$ hit each other. By Lemma 40, this contradicts $(S_u \cup S_v)^c = \emptyset$. Otherwise, $u_b = b_v$, but that is impossible since $u_b$ cannot be reduced in the elimination ordering of $S_u$. □

Lemma 53. If $b_u \in S_u \cap S_v \cap S_w$ and let $u_1, \ldots, u_m$ be the elimination ordering of $S_v$ and $v_1, \ldots, v_m$ be the corresponding out-elimination ordering. Assume there exists $t$ minimal such that $u_t \in S_u \cap S_v^c$. Then, $v_t \notin S_w$ and $u_t$ has out-neighbors in $S_u \cap S_v \cap S_w^c$.

Proof. Consider $N_{out}(u_t)$ and note that $N_{out}(u_t) \subseteq \{u_1, \ldots, u_m, v_t\}$, so we know that $N_{out}(u_t) \subseteq S_v$.

By Lemma 33, since $u_t \neq b_u$, we also know $N_{out}(u_t) \subseteq S_u$. Next, note that $u_t$ cannot only have out-neighbors in $S_w$, or else $u_t \in S_w$. So, $u_t$ must have out-neighbors in $S_u \cap S_v \cap S_w^c$.

Now, if $v_t \in S_w$, then note that $b_w$ could not have been previously reduced by Lemma 51. Therefore, if $v_t$ has any out-neighbors in $S_w$, they cannot have been reduced to its out-neighbors in $S_w^c$. Since $u_t$ has out-neighbors in $S_u \cap S_v \cap S_w^c$, they must have been reduced previously in $u_1, \ldots, u_{t-1}$, contradicting minimality of $t$. □
Lemma 54. Let $G \in \mathcal{F}$. If $b_u \in S_u \cap S_v \cap S_w$, then either $S_u \cap S_v^c \cap S_w = \emptyset$ or $S_u \cap S_v \cap S_w^c = \emptyset$.

Proof. Assume otherwise that both sets are non-empty. Consider the elimination order of $S_u$, $u_1, ..., u_m$ and its corresponding out-sequence $v_1, ..., v_m$, where $u_1 = u$ and $v_m = b_u$. Note that $u_1 \in (S_v \cup S_w)^c$ and we know that since $b_u \notin S_u \cap S_v^c \cap S_w, S_u \cap S_v \cap S_w^c$, there must exist minimal $l > 1$ such that $u_l \in S_v \cup S_w$.

Without loss of generality, we may assume that $u_l \in S_v$. Assume that there exists a minimal $r \geq l$ such that $v_r \notin S_v$. But $(S_u \cap S_v)^c \neq \emptyset$ and $b_u \in S_v$, which contradicts Lemma 52.

So, no such $r$ exists. Now, since $b_u \notin S_u \cap S_w \cap S_v^c$ and it is a non-empty set, we can find a minimal $q > l$ such that $u_q \in S_u \cap S_w \cap S_v^c$. Note $v_q \in S_v$, which contradicts Lemma 53. \hfill \Box

Lemma 55. Let $G$ be a minimal forbidden minor. If $b_u \in S_u \cap S_v \cap S_w$ and $b_u, b_v$ are not, then either $S_u \cap S_v^c \cap S_w = \emptyset$ or $S_u^c \cap S_v \cap S_w = \emptyset$.

Proof. Since $b_u \neq b_v, b_w$, first note that the out-neighbors of $b_u$ in $S_v^c$ are contained in $S_v \cap S_w$ and there are at least 2 such out-neighbors. Assume that both sets are non-empty. Consider the elimination order of $S_w$, $u_1, ..., u_m$, and its corresponding out-sequence $v_1, ..., v_m$, where $u_1 = w, v_m = b_w$. Note that $u_1 \in (S_u \cup S_v)^c, b_u \in S_u \cup S_v$, so we can find minimal $l > 1$ such that $u_l \in S_u \cup S_v$.

Case 1: $u_l \in S_u$. Note that $b_w \notin S_u$, or else $S_u, S_w$ will hit one another. So, there must exist minimal $t \geq l$ such that $v_t \notin S_u$. By Lemma 52, $u_t = b_u$. But $b_u$ has at least two out-neighbors in $S_u^c \cap S_v \cap S_w$ and so one of its out-neighbors must have been previously reduced and there exists a minimal $q < t$ such that $u_q \notin S_u^c \cap S_v \cap S_w$. We see that $q > l$.

So, $u_q \notin S_u, v_q \in S_u$, thus in $G$ $u_q$ has an out-neighbor $x \in S_u^c$ in $G$ that must have been previously reduced. We see that if $x \in S_v$, then the minimality of $y$ is contradicted. We must have $x \notin S_v$ and since $u_q \in S_v, u_q = b_v$ by Lemma 33 and $b_v \in S_w$.

Note that $v_{t-1} = b_u = v_t$ is forced since $v_{t-1} = v_t$ contradicts the minimality of $t$. But, $v_{t-1} = b_u \in S_v$ and $t - 1 \geq q$, where $u_q = b_v$, contradicting Lemma 51.

Case 2: $u_l \in S_v \cap S_v^c$. Since $S_u \cap S_v^c \cap S_w \neq \emptyset$, we see that there exists a minimal $t \geq l$ such that $u_l \in S_v^c \cap S_u$ or $v_l \in S_v^c \cap S_u$.

Case 2.1: $u_l \in S_v^c \cap S_u$. Then by Lemma 53, $u_l$ has an out-neighbor $x$ in $G$ in $S_v^c$. Note either $x$ was previously reduced or exists $k < t$ such that $v_k = x$. Either case, $x \in S_u$ contradicts the minimality of $t$. So $x \notin S_u$, but $u_l \neq b_u$ contradicts Lemma 33.

Case 2.2: $v_t \in S_v^c \cap S_u$. Let $q \leq t$ be minimal such that $v_q \in S_v^c$. By Lemma 52, we see that $u_q = b_v$. If $b_u$ was reduced on $u_l$ or before, then $v_l \in S_v$ contradicts Lemma 51.

Since $b_u \neq b_w$, there must exist a minimal $r > t$ such that $u_r = b_u$. Note that if $v_{r-1} = b_u \in S_v$, then since $b_u$ is already contracted, this contradicts Lemma 51. Therefore, $v_{r-1} = v_r$. However, we claim that $v_{r-1} \in S_u$. Assume not and consider the minimal $t < p < r$ such that $v_p \notin S_u$. Then, we see that $v_{p-1} = u_p$ and so, $u_p \in S_u$. However, since $b_u$ is not yet reduced, this forces $u_p = b_u$, a contradiction to the minimality of $r$. So, $v_r = v_{r-1} \in S_u$. But $u_r = b_u$ and $v_r \in S_u$ contradicts Lemma 51. \hfill \Box

Lemma 56. If $b_u, b_v \in S_u \cap S_v \cap S_w$, then $S_u \cap S_v^c \cap S_w = \emptyset$ or $S_u^c \cap S_v \cap S_w = \emptyset$.

Proof. Note that if $b_u$ is in the triple intersection, then this follows from Lemma 54. So, we may assume $b_u$ is in the triple intersection.

Assume that both sets are nonempty. Consider the elimination ordering of $S_w$ as $u_1, ..., u_m$ and its corresponding out-sequence $v_1, ..., v_m$. Now, since $b_u \in S_u \cap S_v \cap S_w$, we can find a minimal $l$
such that \( u_i \in S_u \cup S_v \). Without loss of generality, let \( u_i \in S_u \). Now, note that \( b_w \in (S_u \cup S_v)^c \), or else we get two sets that hit each other by Corollary 45. So, since \( S_u^c \cap S_v \cap S_w \neq \emptyset \), we can find minimal \( q \geq l \) such that \( u_q \in S_u^c \cap S_v \) or \( v_q \in S_u^c \cap S_v \).

Case 1: \( u_q \in S_u^c \cap S_v \). Then by Lemma 53, \( u_q \) has an out-neighbor \( x \) in \( S_u^c \). Note that the out-neighbor cannot be in \( S_v \) since \( x \) must be previously reduced or reduced to, contradicting minimality. Thus, since \( u_q \neq b \), we have a contradiction.

Case 2: \( v_q \in S_u^c \cap S_v \). By Lemma 52, \( u_q = b_j = b \) and \( v_q \in S_v \), contradicting Lemma 51.

Lemma 57 (Non-Intersection Theorem). If \( S_u \cap S_v \cap S_w = \emptyset \), then \( S_u \cap S_v^c \cap S_w = \emptyset \) or \( S_u \cap S_v \cap S_w^c = \emptyset \).

\[ \text{Proof.} \] If \( b_u \) is in the triple intersection, then it follows by Lemma 54. If \( b_u \) is not in the triple intersection, but both \( b_v, b_w \) are, then it follows by Lemma 56. If both \( b_u, b_v \) are not in the triple intersection, or if \( b_u, b_w \) are not in the triple intersection, then it follows by Lemma 55.

6.5 Applications of Intersection Theorems

Definition 58. \( S_u \) is an A-type (elimination set) if there exists an elimination set \( S_v \) such that \( S_u \) hits \( S_v \). If \( S_u \) is not an A-type, we call it a non A-type.

Lemma 59. If \( S_u, S_v \) are non A-types and \( S_u \cap S_v \neq \emptyset \), then \( b_u, b_v \in S_u \cap S_v \) and \( b_u = b_v \).

\[ \text{Proof.} \] Note that if \( b_u \notin S_u \cap S_v \), then by Corollary 44, \( S_u \) must hit \( S_u \), contradicting that \( S_u \) are non A-types. Therefore, \( b_u, b_v \in S_u \cap S_v \) and by Lemma 43, \( b_u = b_v \).

Lemma 60. If \( S_u \) hits \( S_v \) and \( S_w \) hits \( S_x \) and \( b_u = b_w \), then either \( S_u \) hits \( S_x \) or \( S_w \) hits \( S_v \). Also, \( S_v \cap S_x \neq \emptyset \).

\[ \text{Proof.} \] Assume that \( S_w \) does not hit \( S_v \), then there is an out-neighbor of \( b_w = b_u, v_1 \), such that \( v_1 \notin S_v \cup S_v \). But \( S_u \) hits \( S_v \), so \( v_1 \in S_u \cup S_v \) and therefore we conclude that \( v_1 \in S_u \cap S_v \). Symmetrically, we see that \( v_1 \in S_x \cap S_v \). Therefore, \( v_1 \in S_u \cap S_x \). Now, we split into cases.

Case 1: \( b_u \notin S_x \). By Corollary 44, \( S_x \) hits \( S_u \) and \( b_x \in S_w \). If \( b_x \notin S_v \), then again, \( S_v \) hits \( S_x \). But now, \( S_u, S_v, S_w \) hit each other in a cycle, contradicting that existence of \( S_w \). Lemma 40 So, \( b_x \in S_v \). Now, we see \( b_x \in S_u \cap S_v \) and so by Corollary 44, we must have \( b_u \in S_v \) or else \( S_v \) would hit \( S_u \), a contradiction.

In this case, we have that \( b_u \in S_u \cap S_v \cap S_w \) and \( S_u \) hits \( S_v \). Note that if \( b_u = b_v \), then \( S_u, S_v \) hits each other, contradicting the existence of \( S_w \). Thus \( b_u \neq b_v \), and so \( N_{out}(b_u) \subseteq S_v \). Therefore, \( S_w \) hits \( S_v \), a contradiction.

Case 2: \( b_u \in S_x \). Then, \( b_u = b_w \in S_u \cap S_w \cap S_x \) and since \( S_w \) hits \( S_x \), we can repeat the same argument above to show that \( N_{out}(b_u) \subseteq S_x \). In this case, \( S_u \) hits \( S_x \).

So, in either case there exists an elimination set \( S \) that hits both \( S_v, S_x \). Therefore, we conclude that \( S_v \cap S_x \neq \emptyset \).

\[ \text{Lemma 61.} \] If \( S_u \) hits \( S_v \) and \( S_w \) is a non A-type and \( b_u = b_w \), then \( S_u \cap S_v = \emptyset \).

\[ \text{Proof.} \] Assume that \( S_u \cap S_v \neq \emptyset \), then \( S_v \) cannot hit \( S_u \), or else \( S_u, S_v \) hit each other. So, \( b_u \in S_v \). Therefore, \( b_u \in S_u \cap S_v \cap S_w \) and \( S_u \) hits \( S_v \) and \( S_w \) is a non A-type, so by Lemma 50, we conclude that \( b_u \) is in the triple intersection and \( b_v \) is not. However, since \( b_u = b_v \), by Lemma 49, \( S_w \) must be an A-type that hits \( S_v \), contradicting the fact that \( S_w \) is a non A-type.
7 Minimal Forbidden Minor Properties

Let $G \in \mathcal{F}$. As noted before, $G$ is strongly-connected and $dpw(G) \geq 2$. We say that $S_u$ shares its bad vertex with $S_v$ if $b_u = b_v$.

7.1 A-Types In A Cycle

When there are A-types that hit each other in a cycle, we firstly want to be able to bound the length of the cycle or the number of A-types in the cycle. Using minimality, we show that the number of A-types can be at most 4.

**Theorem 62.** Let $G \in \mathcal{F}$. If there exists a number of A-types that hit each other in a cycle, then there are at most 4 such A-types.

**Proof.** Let $S_{v_1}, \ldots, S_{v_n}$ be A-types such that $S_{v_i}$ hits $S_{v_{i+1}}$, where $v_{n+1} = v_1$, for $1 \leq i \leq n$ and $n > 4$. By Lemma 40, we know that no other vertices or maximal elimination sets exist. Furthermore, we may assume that $n$ is minimal. Therefore, for any $i$, $S_{v_i}$ can only hit $S_{v_{i+1}}$, or else we can find a smaller cycle of A-types that hit each other, a contradiction to minimality. Similarly, $S_{v_{i+1}}$ can only be hit by $S_{v_i}$.

Assume that we have more than 4 A-types. Then, our first claim is for any $i$, $S_{v_m}$ cannot intersect both $S_{v_i}, S_{v_{i+1}}$ for $m \neq i, i+1$. Assume, for contradiction, that $S_{v_m} \cap S_{v_i} \neq \emptyset, S_{v_m} \cap S_{v_{i+1}} \neq \emptyset$.

**Case 1:** $S_{v_i} \cap S_{v_{i+1}} = \emptyset$. By Lemma 44, since $S_{v_m}$ cannot hit $S_{v_{i+1}}$, $b_{v_{i+1}} \in S_{v_m}$. Similarly, $S_{v_i}$ cannot hit $S_{v_m}$, so $b_{v_m} \in S_{v_i}$. Since $S_{v_i} \cap S_{v_{i+1}} = \emptyset$, we deduce $b_{v_m} \not\in S_{v_{i+1}}$. So, $S_{v_{i+1}}$ must hit $S_{v_m}$. If $b_{v_i} \notin S_{v_m}$, then $S_{v_m}$ must hit $S_{v_i}$, so there are at most 3 A-types, a contradiction. Similarly, $S_{v_{i+1}}$ has no out-neighbors in $H^c$, so $G = H$ only has 3 elimination sets, a contradiction.

**Case 2:** $S_{v_i} \cap S_{v_{i+1}} \neq \emptyset$. Then all $S_{v_i}, S_{v_{i+1}}, S_{v_m}$ pairwise intersect and note that if $S_{v_m} \cap S_{v_i} \cap S_{v_{i+1}} = \emptyset$, then by Lemma 46, we have 3 A-types that hit each other, a contradiction. So, $S_{v_m} \cap S_{v_i} \cap S_{v_{i+1}} \neq \emptyset$, and $S_{v_i}$ hits $S_{v_{i+1}}$; so by Lemma 50, either $b_{v_i}$ is in the triple intersection and $b_{v_{i+1}}$ is not, or $S_{v_m}$ hits both $S_{v_i}, S_{v_{i+1}}$. As noted before, the second case cannot happen, so we must have $b_{v_i}$ is in the triple intersection and $b_{v_{i+1}}$ is not. By Lemma 49, we see that we must have either $S_{v_m}$ hits $S_{v_{i+1}}$ or $S_{v_i}$ hits $S_{v_m}$, both contradictions. So, our claim holds.

Our second claim is if $S_{v_i}$ intersects $S_{v_j}$, then $j = i, i-1, i+1$, where $v_0 = v_n, v_1 = v_{n+1}$. Assume that $S_{v_i}$ and $S_{v_j}$ intersects and $j \neq i, i-1, i+1$. Then, we cannot have $S_{v_i}$ hits $S_{v_j}$ or $S_{v_j}$ hits $S_{v_i}$ since $j \neq i, i-1, i+1$. Therefore, by Lemma 44, $b_{v_i}, b_{v_j} \in S_{v_i} \cap S_{v_j}$ and since we have more than 2 elimination sets, $b_{v_i} = b_{v_j}$ must hold. So, by Lemma 60, we know that $S_{v_i}$ hits $S_{v_{i+1}}$, or $S_{v_j}$ hits $S_{v_{i+1}}$, both of which are impossible since $j \neq i, i-1, i+1$.

Finally, we proceed with our proof. First, assume that we can find an A-type $S_{v_i}$, such that $v_i$ is not an in-neighbor of $b_{v_i}$. Without loss of generality, let $S_{v_2}$ be our A-type that satisfies this. Then, note that we may reduce $v_2$ in $S_{v_2}$ to get $S'_{v_2}$ and consider the graph $G' = R(G, v_2)$. It is strongly connected and so we claim that we can appeal to the characterization theorem with $v_0 = S_0 = \emptyset, V_1 = S_{v_1} \cup S'_{v_2}, V_2 = S'_{v_2} \cup S_{v_3}$ and $V_i = S_{v_{i+1}}$ for $3 \leq i \leq n-1$. Note that all $V_i$ are strict subsets of $G$ for $3 \leq i \leq n-1$, trivially. Furthermore, $V_1, V_2$ are also strict subsets since they do not contain the lonely vertices in $S_{v_4}$.

Clearly, we still have $\bigcup_{i=0}^{n-1} V_i = V(G')$. Now, note that $v_2$ and its out-neighbor $w_2$ are lonely vertices in $S_{v_2}$ and are not in any other elimination sets. So, for $i \geq 3$, note that $V_i = S_{v_{i+1}}$ does not have $v_2, w_2$ as out-neighbors. This is because if $S_{v_{i+1}}$ does, then since $S_{v_{i+2}}$ hits $S_{v_{i+2}}$, we see
that either \(v_2, w_2 \in S_{u_{i+1}} \cup S_{v_{i+2}}\). But, by the loneliness of \(v_2, w_2\), one of \(S_{u_{i+1}}, S_{v_{i+2}}\) must be \(S_{v_2}\), which is impossible.

So, in \(G', b_{v_{i+1}} \in V_i\) is still the unique vertex with at least 2 edges from \(v_i\) to \(V_i^c\). For \(V_1\), we claim that \(b_2\) also satisfies this constraint. Note that in \(G'\), there are two out-neighbors of \(b_2\) that are in \(S_{v_1}\), however, the problem is that they could be in \(S_{v_2}\). Assume \(w\) is an out-neighbor of \(b_2\) in \(S_{v_1}\), then note that \(w\) must be in \(S_{v_3}\) since \(S_{v_2}\) hits \(S_{v_3}\). But, \(S_{v_1}\) cannot intersect \(S_{v_3}\) when there are more than 3 A-types, by our second claim. Similarly, this holds for \(V_2\).

Lastly, we claim that property 4 holds. If \(u \in V_i \) and there exists \(v \in V_i \) such that \((u, v) \in E(G')\), then in \(G\), we know that \(v \in S_{v_3}\) or \(v \in S_{v_2}\), and \(u \notin S_{v_1}, S_{v_2}\).

**Case 1:** \(v \in S_{v_1}\). Then, we can find \(w \in S_{v_1}^c\) such that \((u, w) \in E(G')\). If \(w \notin S_{v_2}\), then note that \(w \notin S_{v_1} \cup S_{v_2}\) and \((u, w) \in E(G')\), so we are done.

If \(w \in S_{v_2}\), then we see that since \(S_0 = \emptyset\), we claim that \(u\) must be a bad vertex of another A-type, \(S_{v_k}\) and clearly \(S_{v_k} \neq S_{v_1}, S_{v_2}\). If \(u\) is not the bad vertex, then there must be an elimination set \(S_{v_0}\) that contains \(u\) and its out-neighbors. But, \(S_{v_0}\) intersects \(S_{v_1}, S_{v_2}\), a contradiction.

So, there must exist an A-type \(S_{v_k}\) such that \(u = b_{v_k}\). Note that \(S_{v_k}\) hits \(S_{v_{k+1}}\) and \(v, w \in S_{v_k} \cup S_{v_{k+1}}\). If both \(v, w \in S_{v_k}\), then note that \(S_{v_k}\) intersects both \(S_{v_1}, S_{v_2}\), a contradiction to our first claim.

If \(v \in S_{v_k}\), then \(S_{v_k}\) must be \(S_{v_0} = S_{v_n}\), by our second claim. Then, we know that \(S_{v_{k+1}} = S_{v_1}\).

But, \(w \in S_{v_{k+1}} = S_{v_1}\), which contradicts \(w \notin S_{v_1}\).

So we must have \(w \in S_{v_k}\) and by claim 2, we must have \(S_{v_k} = S_{v_3}\). Since \(v \in S_{v_{k+1}}\), we must have that \(S_{v_{k+1}} = S_{v_0} = S_{v_n}\). So, we see that \(n = 4\), contradiction.

**Case 2:** \(v \in S_{v_2}\). This case is symmetric to case 1.

Therefore, \(V_1\) satisfies property 4. Similarly, we may repeat the same argument for \(V_2\).

For \(i \geq 3\), if \(u \in V_i^c\) and there exists \(v \in V_i\) such that \((u, v) \in E(G')\), then note that \((u, v) \in E(G)\) so we can find \(w \in V_i^\complement\) such that \((u, w) \in E(G)\). We see that \(w \in V_i^c\) in \(G'\) and \((u, w) \in E(G')\), unless \(w = v_2\) and \(u = w_2\). Note that by choice, \(u = w_2\) is not the bad vertex, so we must have \(v \in S_{v_2} \cap V_i\) in \(G\), contradicting claim 2.

So, all properties are satisfied and we may use the characterization theorem and get a contradiction to minimality. Therefore, \(G\) has at most 4 A-types that hit each other in a a cycle.

Note that if we cannot find an A-type \(S_{v_i}\) such that \(v_i\) is not an in-neighbor of \(b_{v_i}\), then all \(b_{v_i}\) is lonely. By **Lemma 42**, all \(S_{v_i}\) are lonely. Now, we may repeat the argument above, except during the last paragraph:

For \(i \geq 3\), if \(u \in V_i^c\) and there exists \(v \in V_i\) such that \((u, v) \in E(G')\), then note that \((u, v) \in E(G)\) so we can find \(w \in V_i^c\) such that \((u, w) \in E(G)\). We see that \(w \in V_i^c\) in \(G'\) and \((u, w) \in E(G')\), unless \(w = v_2\) and \(u = w_2 = b_{v_2}\). However, \(N_{out}(b_{v_2}) \subseteq S_{v_2} \cup S_{v_3}\), we have that \(V_i\) either intersects \(S_{v_2}\) or \(S_{v_3}\), contradicting that \(V_i\) is lonely.

\(\square\)

### 7.2 Connected Sets

Recall that \(S_u\) shares its bad vertex with \(S_v\) if \(b_u = b_v\). Note that \(S_u\) shares its bad vertex with itself.

**Definition 63.** An elimination set \(S_u\) is connected to a non A-type \(S_v\) with a connection of length \(L\) if there exists elimination sets \(S_{v_1}, \ldots, S_{v_{L+1}}\) such that \(S_u = S_{v_1}\) and \(S_v\) shares its bad vertex with \(S_{v_{L+1}}\) and for \(1 \leq i \leq L\), \(S_{v_i}\) shares its bad vertex with an elimination set \(S_{w_i}\) such that \(S_{w_i} \cap S_{v_{i+1}}\).
It is important to note that an elimination set can only be connected to a non A-type elimination set. Connection to an A-type elimination set is not defined and is omitted for simplicity. Note that an elimination set \( S_u \) could have multiple connections to multiple non A-types.

If \( S_u \) is a non A-type, then \( S_u \) is connected to itself with length 0. Also, any A-type \( S_u \) is connected to some non A-type \( S_v \) because there are no cycle of A-types that hit each other. Indeed, if \( S_u \) hits a non A-type \( S_v \), then \( S_u \) is connected to \( S_v \). If not, \( S_u \) hits another A-type, which must hit another elimination set. Eventually an A-type must hit a non A-type, \( S_v \) and so \( S_u \) is connected to \( S_v \). A non A-type elimination set is connected to itself and possibly others if it shares its bad vertex with another elimination set.

**Observation 64.** For any elimination set \( S_u \), there exists \( S_v \) such that \( S_u \) is connected to \( S_v \).

**Definition 65.** Let \( S_u \) be a non A-type in \( G \) and let \( S_{v_1},...,S_{v_k} \) be all elimination sets connected to \( S_u \). The **connected set** of \( S_u \) is \( C_u = \bigcup_{i=1}^{k} S_{v_i} \).

If \( S_u \) is connected to \( S_v \) and \( S_v \) is connected to \( S_w \), then \( S_u \) is connected to \( S_w \) by simply concatenating the connections. Similarly, if \( S_u \) hits \( S_v \) and \( S_v \) is connected to \( S_w \) with length \( L \), then \( S_u \) is connected to \( S_w \) with length \( L + 1 \). So these “connections” define a quasi-order, i.e., they are reflexive and transitive.

**Lemma 66.** If \((u, v) \in E(G)\) and \(u \in C_w, v \in C_w^c\), then \( u = b_w \).

*Proof.* Assume that \((u, v) \in E(G)\) and \(u \in C_w, v \in C_w^c\). There exists some \( S_x \) such that \( u \in S_x \subseteq C_w \). By Lemma 33, we see that \( u = b_x \) is forced. If \( S_x \) is connected to \( S_w \) with length 0, then \( b_x = b_w \). Therefore, \( u = b_x = b_w \).

Otherwise, \( S_x \) is connected to \( S_w \) with length \( \geq 1 \), so there exists \( S_{v_1},...,S_{v_{L+1}} \) such that \( S_x = S_{v_1} \) and \( S_u \) shares its bad vertex with \( S_{v_{L+1}} \) and for \( 1 \leq i \leq L \), \( S_{v_i} \) shares its bad vertex with an elimination set \( S_{w_i} \) such that \( S_{w_i} \) hits \( S_{v_{i+1}} \). By definition, it is clear that \( S_{w_1},S_{v_2} \) also is connected to \( S_w \) and so \( S_{w_1},S_{v_2} \subseteq C_w \). Since \( b_{w_1} = b_x \) and \( S_{w_1} \) hits \( S_{v_2} \), we conclude that \( N_{out}(b_x) \subseteq S_{w_1} \cup S_{v_2} \subseteq C_w \), contradicting \( v \in C_w^c \).

**Lemma 67.** If an A-type \( S_u \) is connected to \( S_v \) with length at least \( L \geq 1 \) and \( S_u \) hits \( S_w \), then \( S_w \subseteq C_v \) and \( S_w \) is connected to \( S_v \) with length \( \leq L \).

*Proof.* Since \( S_u \) is connected to \( S_v \) with length \( L \geq 1 \), we see that there are maximal elimination sets \( S_{v_1},...,S_{v_{L+1}} \) such that \( S_u = S_{v_1} \) and \( S_u \) shares its bad vertex with \( S_{v_{L+1}} \) and for \( 1 \leq i \leq L \), \( S_{v_i} \) shares its bad vertex with an elimination set \( S_{w_i} \) such that \( S_{w_i} \) hits \( S_{v_{i+1}} \).

First note that \( S_u \) shares its bad vertex with \( S_{w_1} \) that hits \( S_{v_2} \), where \( S_{v_2} \) shares its bad vertex with \( S_{w_2} \). Since \( S_u \) hits \( S_w \) and \( b_u = b_{w_1} \), by Lemma 60, \( S_w \cap S_{v_2} \neq \emptyset \). Then, if \( S_w \) hits \( S_{v_2} \), then \( S_w \) is connected to \( S_v \) with length \( L \). \( S_w \subseteq C_v \). So, we conclude that \( b_{v_2} \in S_w \).

Now, we will proceed by induction on \( L \). For the base case \( L = 1 \), note \( b_{v_2} = b_v \), so \( S_v \cap S_w \neq \emptyset \). Since \( S_v \) is a non A-type, by Corollary 44, we must have \( b_w \in S_v \). Now, by Corollary 45, we conclude that \( b_{w} = b_v \) and \( S_w \subseteq C_v \) with length \( L - 1 \).

Now, we induct on \( L \) and assume that our result holds for \( L = i \geq 1 \). Then, let \( L = i + 1 \). Note that if \( S_{v_2} \) hits \( S_w \), then we are done by the inductive hypothesis since there is a connection from \( S_{v_2} \) to \( S_v \) with length \( i \). Otherwise, \( S_{v_2} \) does not hit \( S_w \), so \( b_w \in S_{v_2} \). By Corollary 45, \( b_w = b_{v_2} = b_{w_2} \) and so, \( S_w \subseteq C_v \) and \( S_v \) is connected to \( S_w \) with length \( L \).

**Lemma 68.** Let \( S_u \) be any elimination set and \( S_u \cap C_v \neq \emptyset \), then \( S_u \subseteq C_v \).
Proof. If \( S_u \cap C_v \neq \emptyset \), then there exists some elimination set \( S_w \subseteq C_v \) such that \( S_w \cap S_u \neq \emptyset \). If \( S_u \) hits \( S_w \), then \( S_u \subseteq C_v \). Otherwise, \( b_w \in S_u \). Also, if \( b_u = b_w \), then \( S_u \subseteq C_v \). Therefore, we conclude that \( b_u \notin S_w \) and \( S_w \) hits \( S_u \). Now, we do casework based on whether \( S_w \) shares its bad vertex as \( S_v \).

Case 1: \( b_w = b_v \). Since \( S_v \) is a non A-type and \( S_w \) hits \( S_u \), by Lemma 61, \( S_w \cap S_u = \emptyset \), a contradiction.

Case 2: \( b_w \neq b_v \). Then, \( S_w \) is connected to \( S_v \) with length at least 1 and \( S_w \) hits \( S_u \), so \( S_u \subseteq C_v \) by Lemma 67.

Lemma 69. Let \( C_u, C_v \) be connected sets and \( C_u \cap C_v \neq \emptyset \), then either \( C_u \subseteq C_v \) or \( C_v \subseteq C_u \).

Proof. Let \( w \in C_u \cap C_v \), then by Lemma 68, note that any elimination set that contains \( w \) will be in \( C_u, C_v \). So, we can find some elimination set \( S_x \subseteq C_u, C_v \). As before, note that either \( b_x = b_u \) or \( b_x \neq b_u \).

Case 1: \( b_x = b_u \). Then \( S_u \cap C_v \neq \emptyset \) and so, \( S_u \subseteq C_v \). Since \( S_u \) is connected to \( S_v \), we conclude that any set in \( S_u \) is also connected to \( S_v \), by transitivity. So, \( C_u \subseteq C_v \).

Case 2: \( b_x \neq b_u \). Note that \( S_x \) must share its bad vertex with some \( S_y \) that is an A-type connected to \( S_u \). But then, \( S_u \cap C_v \neq \emptyset \) and \( S_u \subseteq C_v \). If \( S_u \) shares a bad vertex with \( S_v \), then repeating the argument in case 1 gives \( C_v \subseteq C_u \).

Otherwise, \( S_u \) is an A-type of length at least 1 in both \( C_u, C_v \), so it hits \( S_y' \) and by Lemma 67, \( S_y' \subseteq C_u, C_v \). Then, repeat the argument until we see either \( S_u, S_v \) and deduce that either \( C_u \subseteq C_v \) or vice versa.

7.3 Minimally Connected Sets

Definition 70. Let \( C_u \) be the connected set of \( S_u \) and let \( C_{v_1}, \ldots, C_{v_k} \) be all the properly contained connected sets of \( C_u \). The **minimally connected set** of \( S_u \) is \( M_u = C_u \cap \bigcap_{i=1}^{k} C_{v_i} \).

Note that by definition, if \( M_{u_1}, \ldots, M_{u_k} \) are all the minimally connected sets and \( C_{u_1}, \ldots, C_{u_k} \) are all the connected sets, then

\[
\bigcup_{i=1}^{k} M_{u_i} = \bigcup_{i=1}^{k} C_{u_i} = S_0^\emptyset
\]

The last equality holds because every elimination set is connected to some elimination set.

Lemma 71. If \( M_u \cap M_v \neq \emptyset \), then \( M_u = M_v \).

Proof. Note that \( C_u \cap C_v \neq \emptyset \). By Lemma 69, either \( C_u \subseteq C_v \) or \( C_v \subseteq C_u \). Without loss of generality, \( C_u \subseteq C_v \). If \( C_u \subseteq C_v \), then \( M_u \cap C_u = \emptyset \), a contradiction. Therefore, \( C_u = C_v \) and by definition, \( M_u = M_v \). □

Minimally connected sets that are not equal do not intersect at all, allowing us to isolate the effects of minor operations in one minimally connected set.

Definition 72. If \( S_u \) is connected to a non A-type \( S_v \), then the **distance** from \( S_u \) to \( S_v \) is the minimum \( L \geq 0 \) such that \( S_u \) has a connection of length \( L \) to \( S_v \). We denote this by \( d(S_u, S_v) \).

Definition 73. \( S_u \) is **minimally connected** to \( S_v \) if there does not exist \( S_w \) such that \( d(S_u, S_w) < d(S_u, S_v) \).
Lemma 74. If $S_u$ is minimally connected to $S_v$ and $d(S_u, S_v) > 0$, then there are no non A-types that share the same vertex as $S_u$.

Proof. If $S_w$ is a non A-type and $b_w = b_u$, then $d(S_u, S_w) = 0$ is a contradiction of minimality.

Lemma 75. If $S_u, S_v$ are connected to $S_w$ and $d(S_u, S_w) - d(S_v, S_w) \geq 2$, then $S_u \cap S_v = \emptyset$.

Proof. Assume otherwise that $S_u \cap S_v \neq \emptyset$. Then, by Lemma 43, either $S_u$ hits $S_v$ or $S_v$ hits $S_u$ or $b_u = b_v$. If $S_u$ hits $S_v$, then $d(S_u, S_w) \leq d(S_v, S_w) + 1$, a contradiction. If $S_v$ hits $S_u$, then by Lemma 67, $d(S_u, S_w) \leq d(S_v, S_w)$. Note that if $S_u$ has distance 0, then $S_u$ cannot intersect $S_v$, by Lemma 60. If $b_u = b_v$, then $d(S_u, S_w) = d(S_v, S_w)$, a contradiction.

Lemma 76. If $S_u \subseteq C_v \subseteq C_w$, then $d(S_u, S_v) < d(S_u, S_w)$.

Proof. Note $C_v \subseteq C_w$ implies that $d(S_v, S_w) > 0$, or else $S_v, S_w$ would share the same vertex and $C_v = C_w$. So, $S_u$ has a connection of length $L = d(S_u, S_v) > 0$ to $S_w$, and there exists $S_{v_1}, \ldots, S_{v_{L+1}}$ such that $S_{v_i} = S_{v_1}$ and $S_w$ shares its bad vertex with $S_{v_{L+1}}$ and for $1 \leq i \leq L$, $S_{v_i}$ shares its bad vertex with an elimination set $S_{w_i}$ such that $S_{w_i}$ hits $S_{v_{i+1}}$.

But by Lemma 66, the only out-neighbors of $C_v$ is from $b_u$ and since $S_w \not\subseteq C_v$, there must exists $k \leq L$ such that $b_v \in S_{v_k}$. So, $S_u$ is connected to $S_{v_k}$ with smaller length and $S_{v_k}$ is connected to $S_v$ with length 0, so $d(S_u, S_v) < d(S_u, S_w)$.

Lemma 77. $M_u$ is the set of all minimally connected sets to $S_u$.

Proof. Let $S_v$ be minimally connected to $S_u$. If it is in $C_w \not\subseteq C_u$, then by Lemma 76, we have a contradiction of minimality. Therefore, all minimally connected sets to $S_u$ are in $M_u$.

Let $S_u \subseteq M_u$, then note that it must be connected to $S_u$. If it is not minimally connected to $S_u$ and is minimally connected to another elimination set $S_w$, then $C_w \not\subseteq C_u$. But, this contradicts $S_u \subseteq M_u$, so $M_u$ is in the set of all minimally sets to $S_u$.

Therefore, we note that if $S_w \subseteq M_u$ is a non A-type, then $d(S_w, S_u) = 0$. And, we see that $b_w = b_u$ is implied. We now consider an equivalence relation on all minimally connected sets $M_{v_1}, \ldots, M_{v_k}$ using set equivalence avoiding duplicates. The choice of the representatives of these equivalence classes is arbitrary.

Under the equivalence relation quotient, by Lemma 71 and Lemma 77, note that none of the minimally connected sets intersect. Furthermore, it is easy to see that if $(u, v) \in E(G)$ and $u \in M_w, v \in M_w$, then $u = b_w$, as for connected sets.

8 Finiteness of Minimal Forbidden Minors

8.1 Finiteness of Minimally Connected Sets

Definition 78. Let $M_u$ be a minimally connected set. Define the extra set of $M_u$, $T_u \subseteq S_0$, to be the smallest subset of $S_0$ such that there does not exist $v \in S_0 \cap T_u$ such that $N_{out}(v) \subseteq M_u \cup T_u$.

Note that this is a recursive definition of $T_u$ and $T_u$ is not the maximal/maximum set in $S_0$ that hits $M_u$. However, it is true that $T_u$ hits $M_u$. If there are no vertices in $S_0$ with all of its out-neighbors in $M_u$, then $T_u = \emptyset$. Since $T_u$ hits $M_u$, we have that if $(w, v) \in E(G)$ and $w \in M_u \cup T_u, v \in (M_u \cup T_u)^c$, then $w = b_u$, as before.
Lemma 79. \((M_u \cup T_u) \cap (M_v \cup T_v) = \emptyset\) for \(u \neq v\).

Proof. Let \(H = (M_u \cup T_u) \cap (M_v \cup T_v)\). If \(H\) is non-empty, then it must have out-neighbors in \(H^c\). So, by our above observation, either \(b_u \in M_v \cup T_v\) or \(b_v \in M_u \cup T_u\). But since \(M_u \cap M_v = \emptyset\), we either have \(b_u \in T_v\) or \(b_v \in T_u\). However, \(T_u, T_v \subseteq S_0\), a contradiction.

Lemma 80. Let \(S_u\) be an non-\(A\)-type and \(H\) be the set of all elimination sets that share the same bad vertex as \(S_u\). Then, \(S_u\) has at most 2 out-neighbors in \(H^c\).

Proof. Assume otherwise and let \(w_1, w_2, w_3 \in H^c\) be three out-neighbors of \(S_u\). Consider deleting \((b_u, w_1)\) to obtain \(G^c\). Note that \(G^c\) might not be strongly connected. If so, we can find \(A, B \subseteq V(G^c)\) such that \(w_1 \in A, b_u \in B\) and all edges go from \(A\) to \(B\). Note that if \(v \in B\), then \(N_{out}(v) \subseteq B\).

Now, consider \(G'' = G'[B]\), which is strongly connected. If \(S_{u_1}, \ldots, S_{u_n}\) is the elimination decomposition of \(G\), then we claim that we can appeal to the characterization theorem with \(V_0 = S_0 \cap B\) and \(V_j = S_{u_j} \cap B\). It is clear that \(\bigcup_{i=0}^{n} V_i = V(G'')\).

Next, we claim that \(V_j \neq G''\), \(j \geq 1\). If \(V_j = G''\), then \(b_u \in V_j\), and so \(b_u \in S_{u_j}\) in \(G\). If \(b_{u_j} \neq b_u\), then \(S_u\) is an \(A\)-type, a contradiction, so we must have \(b_u = b_{u_j}\). But, this means that there are at least two out-neighbors from \(b_{u_j}\) to \(S_{u_j}^c\) in \(G\) and in \(G^c\), we have only deleted one edge, so there must still be an out-neighbor of \(b_{u_j}\) in \(S_{u_j}^c\). Note that the out-neighbor is in \(B\), since it is an out-neighbor of \(b_u \in B\). Therefore, it is in \(G'' \cap S_{u_j}^c\), contradicting \(V_j = G''\). We conclude that \(V_j\) are still strict subsets.

We see that if \(v \in S_0\), then \(v\) still has outdegree \(\geq 2\) in \(G\). If \(v \in G''\), then its outdegree is not reduced, by our previous observation. Thus, \(V_0\) is a set of vertices with outdegree \(\geq 2\).

Now, if \(V_j\) does not contain \(b_{u_j}\), then \(V_j = \emptyset\) since otherwise \(V_j\) has no out-neighbors in \(G'' \cap V_j^c\). For property 3, note that if \(b_{u_j} \neq b_u\) and \(b_{u_j} \in B\), then \(V_j\) still has at least two out-neighbors to \(V_j^c\) as \(N_{out}(b_{u_j}) \subseteq B\). If \(b_u = b_{u_j}\), then \(u_2, u_3 \in V_j\) since \(u_2, u_3 \notin H\) and \(S_{u_j} \subseteq H\). So property 3 holds.

Now, for property 4, if \(w_j \in V_j^c\) and \(x_j \in V_j\) such that \((w_j, x_j) \in E(G'')\), then we see that in \(G\), there existed \(y_j \in V_j^c\) such that \((w_j, y_j) \in E(G)\). Note that if \((w_j, y_j) \in E(G'')\), then since \(w_j \in B, y_j \in B\), so \((w_j, y_j) \in E(G'')\). This does not hold only if \(w_j = b_u\) and \(y_j = u_1\). Note that if \(u_2\) or \(u_3\) is not in \(V_j\), then property 4 still holds. We only run into a problem if we can find an elimination set \(S_{v_1}\) such that \(u_1 \in S_{v_1}^c\) and \(u_2, u_3 \in S_{v_1}\).

Now, consider deleting \((b_u, u_2), (b_u, u_3)\), and we can find \(S_{v_1}, S_{v_2}, S_{v_3}\) so that \(u_2 \notin S_{v_2}, u_1, u_3 \in S_{v_2}\) and \(u_3 \notin S_{v_3}, u_1, u_2 \in S_{v_3}\). It is clear that \(S_{v_1}, S_{v_2}, S_{v_3}\) mutually intersect. We see that \(S_{v_1} \cap S_{v_2} \cap S_{v_3} \neq \emptyset\) or else, by Lemma 46, they would form 3 \(A\)-types that hit each other, contradicting the existence of \(S_u\). But \(u_1 \in S_{v_1} \cap S_{v_2} \cap S_{v_3}\) and \(u_2 \in S_{v_1} \cap S_{v_2} \cap S_{v_3}\) contradicts Lemma 57.

So property 4 holds, and \(G''\) is a minor of \(G\) with directed pathwidth \(\geq 2\), contradicting the minimality of \(G\). \(\square\)

Corollary 81. Let \(M_u\) be a minimally connected set. \(M_u\) has at most 2 out-neighbors to \(M_u^c\).

Lemma 82. If \(M_u\) has at least 2 out-neighbors in \((M_u \cup T_u)^c\), then there exists another \(M_v\) that has at least 2 out-neighbors in \(M_u \cup T_u\).

Proof. Consider reducing \(u\) in \(M_u\) to get \(M_u'\) and we claim that our resulting graph \(G'\) is still of directed pathwidth 2, unless we can find a minimally connected set \(M_v\) with at least two out-neighbors in \(M_u \cup T_u\).
Assume for the sake of contradiction that such a $M_v$ does not exist. Let $S_{u_1}, \ldots, S_{u_s}$ be the elimination decomposition of $G$. We try to apply characterization theorem with $V_0 = S_0 \cap T_u^c$ and $V_j = S_{u_j} \cap M_{u_j}^c$, for $1 \leq j \leq s$. Lastly, we let $V_{s+1} = M_{u_s}^c \cup T_u$. Clearly, $\bigcup_{i=0}^{s+1} V_i = V(G')$.

Note that all $V_j$ are strict subsets as before and $V_{s+1}$ is a strict subset since $M_u \cup T_u$ has out-neighbors in $(M_u \cup T_u)^c$ in $G$ and thus, $V_{s+1}$ is a strict subset in $G'$. Furthermore, note that $S_0$ still have vertices of outdegree $\geq 2$ in $G'$.

Let $w$ be the out-neighbor of $u$. For property 3, for $1 \leq j \leq s$, if $V_j = S_{u_j} \cap M_{u_j}^c$ is non-empty and has lost out-neighbors in $V_j^c$ after the contraction of $u$, then $V_j$ must have $u, w$ as its out-neighbors in $G$. We see that $V_j \subseteq M_{u_j}^c$ and so there must exist some $M_v$ such that $V_j \subseteq M_v$. Clearly, $V_j$ has at least two out-neighbors in $M_u \cup T_u$ and this contradicts our assumption that no such $M_v$ exists. So, we see that property 3 holds. For $V_{s+1}$, property 3 holds by assumption.

For property 4, for $1 \leq j \leq s$, realize in $G$, if $x \in S_{u_j}^c$ and there exists $y \in S_{u_j}$ such that $(x, y) \in E(G)$, then there exists $z \in S_{u_j}^c$ such that $(x, z) \in E(G)$. When $V_j = S_{u_j} \cap M_{u_j}^c$ is non-empty, note that unless we have $x = w$ and $z = u$, $(x, z) \in E(G')$ holds or even if $z = u$, then $(x, w) \in E(G')$ and $w \in S_{u_j}^c$ holds. Since $V_j \subseteq M_{u_j}^c$, it follows that $x = w = b_u$.

If $b_u$ has all of its out-neighbors in $S_{u_j}$ in $G'$, then we see that in $G$, $N_{out}(b_u) \subseteq S_{u_j} \cup \{u\}$. This implies that $S_{u_j}$ hits $S_{u_j}$, contradicting that $S_{u_j}$ is a non A-type.

For $V_{s+1} = M_{u_s}^c \cup T_u$ and if $x \in V_{s+1}^c$ and there exists $y \in V_{s+1}$ such that $(x, y) \in E(G)$, then we consider the following cases.

**Case 1:** If $x \in S_0$, then we can find $z \in V_{s+1}^c$ such that $(x, z) \in E(G')$ unless and $N_{out}(x) \subseteq V_{s+1}$. However, this implies that $x \in T_u$, a contradiction.

**Case 2:** If $x \in M_v$, then $x = b_u$. Note that in $G$, we can find $z \neq x$ such that $(x, z) \in E(G)$. If $z \not\in M_u \cup T_u$, then $z \not\in V_{s+1}$ and $(x, z) \in E(G')$, so property 4 holds. Otherwise, $z \in M_u \cup T_u$ and $M_v$ has at least two out-neighbors $y, z \in V_{s+1}$, but this contradicts our assumption.

Now, by the characterization theorem, $G'$ having pathwidth $\geq 2$ contradicts the minimality of $G$. Therefore, we must be able to find $M_v$ such that $M_v$ has at least two out-neighbors in $M_u \cup T_u$.

**Lemma 83.** If $M_v$ has at least 2 out-neighbors in $(M_u \cup T_u)^c$, then there are exactly two minimal connected sets and its extra set and they hit each other.

**Proof.** By Lemma 82, we can find $M_u$ such that it has at least two out-neighbors in $M_u \cup T_u$. By Lemma 79, $M_u \cup T_u \subseteq (M_{u_1} \cup T_{u_1})^c$ and by Lemma 80, $M_u \cup T_u$ has exactly 2 out-neighbors in $M_u \cup T_u$. In fact, the outdegree restriction implies $M_{u_1} \cup T_{u_1}$ hits $M_u \cup T_u$. And appealing to 82 again, we can find $M_{u_2}$ such that it has at least two out-neighbors in $M_{u_1} \cup T_{u_1}$.

Let $M_u = M_{u_0}$. We can keep doing this to find a sequence of $M_{u_1}, M_{u_2}, \ldots$ and since the graph is finite, there must exist $0 \leq m < n$ such that $M_{u_m} = M_{u_n}$. We recall that $M_{u_1} \cup T_{u_1}$ hits $M_{u_{i+1}} \cup T_{u_{i+1}}$ in this sequence for $i \geq 0$. So, $H = \bigcup_{k=m}^{n-1} (M_u \cup T_u)$ has no out-neighbors in $H^c$.

So, our graph $G$ is made up $t$ different minimally connected sets and their extra sets that hit each other in a cycle. Assume that $t > 2$ and relabel our minimally connected sets to be $M_{u_1}, \ldots, M_{u_t}$. Let $S_{u_1}, \ldots, S_{u_t}$ be the elimination decomposition of $G$.

Consider the contraction $v_2$ in $M_{u_2}$ to get $M_{u_2}^c$. Now, we appeal to characterization theorem in the resulting graph $G'$ with $V_0 = S_0 \cap T_{u_2}^c$ and for $1 \leq j \leq s$

\[
V_j = \begin{cases} S_{u_j} \cap (M_{u_2}^c) & b_{u_j} \neq b_{u_1} \\ S_{u_j} \cup M_{u_2}^c \cup T_{u_2} & b_{u_j} = b_{u_1} \end{cases}
\]
It is clear that $\bigcup_{i=0}^{s} V_i = V(G')$. Note that these sets $V_j$ are all strict subsets of $V(G')$ since each $S_{v_j}$ is in one of $M_{u_1}, ..., M_u$ and $t > 2$, so there must exists some $M_v$ such that $V_j \not\subseteq M_v$.

For property 3, for $1 \leq j \leq s$, if $v_j \not\in S_{v_{j-1}}$, then $v_j$ does not have out-neighbors in $M_{u_2} \cup T_{u_2}$, so $V_j = S_{v_j} \cap (M_{u_2})^c$ still has two out-neighbors in $V_j^c$. If $v_j = b_{u_1}$, then $V_j = S_{v_j} \cup (M_{u_2}) \cup T_{u_2}$ and note that $b_{u_2}$ has at least two out-neighbors in $M_{u_3} \cup T_{u_3}$. Therefore, property 3 holds.

For property 4, for $1 \leq j \leq s$, if $v_j \not\in S_{v_{j-1}}$, then if $(x, y) \in E(G')$ with $x \in V_j^c, y \in V_j$, then we can find $z \in S_{v_j}$ such that $(x, z) \in E(G)$. Note that $(x, z) \in E(G')$ unless $z = w_2$. Let $w_2$ be the out-neighbor of $u_2$. If $x \neq w_2$, then $(x, w_2) \in E(G')$ and $w_2 \in V_j^c$ since $S_{v_j} \subseteq M_{u_2}^c$. If $x = w_2, z = w_2$, then $x = w_2 = b_{u_2}$ is forced. But, $N_{\text{out}}(w_2) \subseteq S_{v_j} \cup \{w_2\}$ implies that $S_{v_j}$ hits $S_{v_j}$, contradicting that $S_{v_j}$ is a non-A-type. So, we can find $z \in V_j^c$ with $(x, z) \in E(G')$.

If $v_j = b_{u_1}$, then we claim that $y \in S_{v_j}$. If $y \notin S_{v_j}$, then $y \in M_{u_2} \cup T_{u_2}$. But since only $M_{u_1} \cup T_{u_1}$ has out-neighbors in $M_{u_2} \cup T_{u_2}$, we must have $x \in M_{u_1} \cup T_{u_1}$, implying $x = b_{u_1}$, a contradiction since $x \in V_j^c$. Therefore, $y \in S_{v_j}$ and we can find $z \in S_{v_j}$ such that $(x, z) \in E(G)$. Note that $z \notin S_{v_j} \cup T_{u_2}$ since $x \in M_{u_1}$ forces $x = b_{u_1}$, leading to the same contradiction. Therefore, $z \notin V_j$ and $(x, z) \in E(G')$.

So, $G'$ has directed pathwidth $\geq 2$, contradicting the minimality of $G$. So, $t \leq 2$, but we see that $t \geq 2$ since by Lemma 82, there must exist another $M_v \cup T_v$ that hits $M_u \cup T_u$. So, $t = 2$. \hfill \Box

**Theorem 84.** There are at most 2 minimally connected sets.

**Proof.** If there exists $M_u$ such that $M_u \cup T_u$ has at least 2 out-neighbors in $(M_u \cup T_u)^c$, then by Lemma 83, we have exactly 2 minimally connected set. So, all minimally connected sets $M_u$ has at most 1 out-neighbor in $(M_u \cup T_u)^c$.

Assume that there are more than 2 minimally connected sets. Let $M_u, M_v$ be two different such sets. By strong connectivity, there is a directed path from $b_u$ to $M_u$, denoted as $P(b_u, M_u)$. We see that except for the endpoints, the vertices of $P(b_v, M_v)$ are in $(M_u \cup T_u \cup M_v \cup T_v)^c$. Similarly, we can find a reverse path $P(b_v, M_u)$ with such properties.

If $x, y \in P(b_v, M_u) \cap P(b_u, M_u)$, then note that either there is a directed cycle that includes $x, y$ and we can cycle contract, or if not, then $x \neq b_u, b_v$, so we may just out-contract. By cycle contraction, it worth noting that $b_u$ is now reduced to $b_v$. So, we may assume that the paths share at most one common vertex.

Otherwise, out-contract each vertex in $(M_u \cup T_u \cup M_v \cup T_v)^c$ until each path at most 2 edges (one from $b_u$ to the common vertex and the common vertex to $M_u$). Using the characterization theorem, we claim that this subgraph is still of pathwidth $\geq 2$.

Consider the new graph obtained and let $H = M_u \cup T_u \cup M_v \cup T_v \cup \{c\}$, where $c$ is the possibly non-existent common vertex. Now, let $S_{v_1}, ..., S_{v_s}$ be the elimination sets contained in $M_u, M_v$ and let $V_0 = T_u \cup T_v \cup \{c\}, V_j = S_{v_j}$ for $1 \leq j \leq s$. Clearly, $\bigcup_{i=0}^{s} V_j = V(H)$ and all $V_j$ are strict subsets of $H$.

Next, note that $T_u, T_v$ has no out-neighbors in $(M_u \cup M_v)^c$ in $G$, so they are still of out-degree $\geq 2$. If $c$ exists, then note that $c$ has one out-neighbor in $M_u$ and one in $M_v$. So, $V_0$ has vertices of out-degree $\geq 2$.

It is important to observe that in $G$, $b_u$ has at most 1 out-neighbor in $(M_u \cup T_u)^c$, meaning that $b_u$ has an out-neighbor $w \in V_j \cap (M_u \cup T_u)$. In $H$, note that $b_u$ also has an out-neighbor in $(M_u \cup T_u)^c$ and so, it has two distinct vertices in $V_j^c$.
For property 3, if $V_j = S_{v_j}$ does not satisfy $b_{v_j} = b_u$ or $b_{v_j} = b_v$, then $N_{\text{out}}(S_{v_j})$ is strictly contained in $M_u, M_v$, so $b_{v_j}$ still has two out-neighbors in $S_{v_j}^c$. If $b_{v_j} = b_u$, then our observation above gives us property 3. Similarly if $b_{v_j} = b_v$.

For property 4, if $(x, y) \in E(H)$ and $x \in V_j, y \in V_j^c$, then note that in $G$, there exists $z \in V_j^c$ such that $(x, z) \in E(G)$. If $x \neq b_u, b_v$, then $z \in H$ and $(x, z) \in E(H)$. Otherwise, $x = b_u$ and if $V_j \subseteq M_u \cup T_u$, then in $H$, $b_u$ has an out-neighbor in $(M_u \cup T_u)^c$ by the above observation. If $V_j \subseteq M_v \cup T_v$, then again, $b_u$ has an out-neighbor in $M_u \cup T_u \subseteq (M_v \cup T_v)^c$. Similarly if $x = b_v$.

So, by the characterization theorem, $H$ contradicts the minimality of $G$ and there are at most 2 minimally connected sets.

8.2 Finiteness of Elimination Sets

In previous section we gave a bound on the number of minimally connected sets. Furthermore, we have bound the number of out-neighbors of a minimally connected set (and its extra set) to its complement to be at most 2. From this, we intuit that a minimally connected set cannot contain many elimination sets and these elimination sets cannot have large distance.

In this section, we show that the number of elimination sets is indeed finite. We will show that for any minimally connected set $M_u$, the number of elimination sets in $M_u$ is at most 4. To understand the behavior of $M_u$, we start by understanding the behavior of $S_0$ and the extra sets.

**Theorem 85.** All vertices in $S_0$ have outdegree exactly 2.

**Proof.** Let $S_{u_1}, \ldots, S_{u_s}$ be the elimination decomposition of $G$ and assume that $v \in S_0$ have more than 2 out-neighbors, say $w_1, w_2, w_3$. Consider deleting $(v, w_1)$ to get $G'$. Note that $G'$ might not be strongly connected. If so, we can find $A, B \subseteq V(G')$ such that $w_1 \in A, v \in B$ and all edges go from $A$ to $B$. Note that if $v \in B$, then $N_{\text{out}}(v) \subseteq B$.

Now, consider $G'' = G'[B]$, which is strongly connected. Consider appealing to the characterization theorem with $V_0 = S_0 \cap B$ and $V_j = S_{u_j} \cap B$ for $1 \leq j \leq s$. Clearly, we have $\bigcup_{j=0}^{s} V_j = V(G'')$. Furthermore, for $1 \leq j \leq s$, $V_j$ is a strict subset of $G''$ since $v \in G''$ and $v \notin V_j$. Also, $V_0$ has vertices of outdegree $\geq 2$ since all vertices in $V_0$, including $v$, still retained outdegree $\geq 2$ in $G'$. Then, in $G''$, those vertices in $B$ did not lose outdegree, so $V_0$ still has vertices of outdegree $\geq 2$.

For property 3, if $V_j \neq \emptyset$, note that by Lemma 33, $b_{v_j} \in V_j$. Now, $b_{v_j}$ is still the only vertex in $V_j$ with out-neighbors in $V_j^c$ and all its out-neighbors are in $G''$. For property 4, we want to show that if $(x, y) \in E(G'')$ and $x \in V_j^c, y \in V_j$, then there exists $z \in V_j^c$ such that $(x, z) \in E(G'')$. Note that in $G$, we can find $z \in S_{v_j}^c$ such that $(x, z) \in E(G)$. Unless $x = v, z = w_1, z \in B$ since $x \in B$ and $(x, z) \in E(G'')$.

So, we only have a problem when there exists $S_{v_1}$ such that $N_{\text{out}}(v) \subseteq S_{v_1} \cup \{w_1\}$ and $w_1 \notin S_{v_1}$. Now, consider deleting $(v, w_2), (v, w_3)$ and we can similarly find $S_{v_2}, S_{v_3}$ such that $N_{\text{out}}(v) \subseteq S_{v_2} \cup \{w_2\}$ and $w_2 \notin S_{v_2}, N_{\text{out}}(v) \subseteq S_{v_3} \cup \{w_3\}$ and $w_3 \notin S_{v_3}$.

But, $w_1 \in S_{v_1} \cap S_{v_2} \cap S_{v_3}, w_2 \in S_{v_1} \cap S_{v_2} \cap S_{v_3}, w_3 \in S_{v_1} \cap S_{v_2} \cap S_{v_3}$. Since $S_{v_1}, S_{v_2}, S_{v_3}$ pairwise intersect, by Lemma 46, we must have $S_{v_1} \cap S_{v_2} \cap S_{v_3} \neq \emptyset$, or else the existence of $v$ would be contradicted. But, $w_1, w_2$ contradicts Lemma 57.

We conclude that property 4 must hold for one of $w_1, w_2, w_3$ and deleting the respective edge gives $G'$ with directed pathwidth $\geq 2$, by the characterization theorem. This contradicts the minimality of $G$. 

□
Lemma 86. Let $u, v \in S_0$ and $(u, v) \in E(G)$ and let $G'$ be graph obtained by out-contracting $(u, v)$. Then, if $S'_0 = S_0 \setminus \{u\}$ has vertices of outdegree $\geq 2$ in $G'$ and there does not exist $w \in V(G)$ such that $w$ has $u, v$ as out-neighbors, then $G'$ has directed pathwidth $\geq 2$.

Proof. Let $S_{u_1}, ..., S_{u_6}$ be the elimination decomposition of $G$ and let $G'$ be defined as above. Note that $G'$ might not be strongly connected, in which case, we can find $A, B \subseteq V(G')$ such that $v \in B$.

Now, consider $G'' = G'[B]$ and $V_0 = S'_0 \cap B$ and $V_j = S_{u_j} \cap B$. Note that all the properties hold. The first assumption gives property 2 and the second assumption gives property 3, 4. By the same argument in Theorem 85, the characterization theorem gives us that $G''$ has directed pathwidth $\geq 2$. This also implies that $G'$ has directed pathwidth $\geq 2$. □

Theorem 87. For any $M_u$, $|T_u| \leq 3$. There exists only one vertex $v \in T_u$ such that $N_{out}(v) \cap M_u \neq \emptyset$.

Proof. It is crucial to note that there are no directed cycles in $G[T_u]$, which can be quickly seen from the definition. This is because if $C \subseteq T_u$ has a directed cycle, then let $H \subseteq T_u$ be all the vertices in $H$ with a directed path in $G[T_u]$ to some vertex in $C$. Consider $T'_u = T_u \setminus H$ and notice that $T'_u$ has out-neighbors only in $T'_u \cup M_u$ since no vertex in $T'_u$ has out-neighbors in $H$. Furthermore, if $v \in H$, then $v$ has an out-neighbor in $H$. Therefore, $T'_u$ contradicts the minimality of $T_u$.

Case 1: Assume that we can find a $M_u$ in $G$ such $M_u \cup T_u$ has 2 out-neighbors in $(M_u \cup T_u)^c$. In this case, by Lemma 83, the graph only contains another $M_v \cup T_v$ such that $M_u \cup T_u$ and $M_v \cup T_v$ hit each other. By Lemma 80, $M_u$ has no out-neighbors in $T_u$ and similarly for $M_v$.

If $M_v$ does not have out-neighbors in $T_u$, then $T_u = \emptyset$ by strongly connectivity. Otherwise, $M_v$ has out-neighbors in $T_u$ and since $G[T_u]$ is a directed acyclic graph, let $w \in T_u$ be a vertex with no in-neighbors in $T_u$ and some out-neighbor in $T_u$. Let $x \in T_u$ be an out-neighbor of $w$ in $T_u$.

Consider out-contracting $(w, x)$ in $T_u$ to get $T'_u$. Note that $T_u \setminus \{w\}$ has no out-neighbors to $w$, so only $M_v$ (or $b_v$) has $w$ as an out-neighbor. Therefore, $T'_u$ still has vertices of outdegree $\geq 2$. Consider out-contracting $w$ to its out-neighbor in $T_u$ and since $T'_u$ still has vertices of outdegree $\geq 2$. If $b_v$ does not have $w, x$ as out-neighbors, then Lemma 86 gives a contradiction to the minimality of $G$. We immediately deduce that if $M_v$ only has one out-neighbor in $T_u$, then $|T_u| \leq 1$.

Otherwise, $M_v$ has two out-neighbors in $T_u$, then $x, w$ must be its only out-neighbors in $T_u$. Note that $x$ has a directed path $P_{xy}$ to $y \in T_u$, where $N_{out}(y) \subseteq M_u$ due to the DAG property. By Theorem 85, we see that $w$ has one other out-neighbor $z$.

If $z$ is on the path, then consider out-contracting $P_{xz}$ until $P_{xz}$ becomes only an edge $(x, z)$. Then, out-contract $P_{yz}$ until $z, y$ are identified. out-contraction preserves outdegree of $w$ to be $\geq 2$ since $G[T_u]$ is a directed acyclic graph. Delete any other vertices in $T_u$. Note that $w, x, z$ has outdegree $\geq 2$ and by the characterization theorem, $G$ still has directed pathwidth $\geq 2$. Thus, $|T_u| \leq 3$.

Note that $x$ could have out-neighbors in $M_u$. If $x$ does, then let $y$ be $a$ be its out-neighbor in $M_u$.

First, out-contract $(x, a)$ and next out-contract $(w, z)$ and note that $z$ is the only remaining vertex in $T_u$ and $|T_u| \leq 1$.

If $z$ is not on the path and there is a path $P_{za}$, where $a \in P_{xy}$, then first out-contract $P_{za}$, and then follow the same argument above. Otherwise, there is a path $P_{za}$ that does not intersect $P_{xy}$ such that $a \in M_u$. Then fully out-contract $P_{xy}$ until $x, y$ are identified. Then, fully out-contract $P_{za}$ until $z$ and $a$ are identified. Then, delete anything in $T_u$ until only $x = y$ is left in $T_u$ and it has outdegree $\geq 2$. By the characterization theorem, it can be checked that $G$ still has directed pathwidth $\geq 2$. Thus, $|T_u| \leq 1$.
Case 2: All $M_u \cup T_u$ only has at most 1 out-neighbor in $(M_u \cup T_u)^c$. If there exists a $M_u \cup T_u$ with exactly 1 out-neighbor in $(M_u \cup T_u)^c$, then by Lemma 84, $G$ has at most one other $M_v \cup T_v$ with also exactly 1 out-neighbor in $(M_u \cup T_u)^c$.

Case 2.1: If $M_v$ exists, note that $M_v$ has at most 1 out-neighbor in $T_v$, by Lemma 80, and $M_v$ also has at most 1 out-neighbor in $T_u$. We conclude that there does not exist $w$ with two out-neighbors in $T_u$ and so, the same argument in case 1 holds and we deduce that $|T_u| \leq 1$.

Case 2.2: If $M_v$ does not exist, then note that $H = (M_u \cup T_u)^c$ must be a subset of $S_0$. But by Theorem 85, if $w \in H$, then $w$ has outdegree $\geq 2$ and since $w \notin T_u$, $w$ has an out-neighbor in $H$. We conclude that $w$ does not have two out-neighbors in $T_u$ and so, the same argument in case 1 holds and we deduce that $|T_u| \leq 1$.

Case 2.3: $M_u \cup T_u$ has no out-neighbors in $(M_u \cup T_u)^c$. In this case, $M_u$ has both its out-neighbors in $T_u$ and the same argument in case 1 holds and we deduce that $|T_u| \leq 3$.

Lemma 88. If $b_u, b_v, b_w \in S_u \cap S_v \cap S_w$ and $b_u$ has out-neighbors in $S_u \cap S_v^c \cap S_w^c$ and $S_u^c \cap S_v \cap S_w^c$, then $S_u, S_v, S_w$ are the only elimination sets in $G$.

Proof. Consider $H = S_w \cap S_u^c \cap S_v^c$, which is not empty, since $v_k \in H$. By strong connectivity, $b_u$ has a directed path to $H$. Note that if the path is completely contained in $I = S_u \cup S_v \cup S_w$, then there exists $x \in I \cap H^c, y \in H$ such that $(x, y) \in E(G)$. Since $x \in S_u \cup S_v$ and $y \in S_u^c \cap S_v^c$, it follows that $y = b_u$. Then, $I$ is a minimal minor by using the characterization theorem on $S_u, S_v, S_w$.

Otherwise, the simple path from $b_u$ to $H$ is in $I^c$, except for the endpoints. We can out-contract all vertices in $I^c$ along the path and deduce the resulting graph to have directed pathwidth $\geq 2$.

Lemma 89. Let $G \in \mathcal{F}$. If there are at least two bad vertices in $G$ or $S_0 \neq \emptyset$, then there cannot exist $S_u$ such that $b_u$ has more than 1 out-neighbor in $S_u$.

Proof. Let $G$ be a minimal forbidden minor with at least 2 bad vertices or $S_0 \neq \emptyset$ and assume for contradiction there exists $S_u$ be a maximal elimination set such that $b_u$ has at least 2 out-neighbors in $S_u$, say $v, w$. Let $H$ be the vertex set that contains all $S_x$ such that $b_x = b_u$. Since there are at least 2 bad vertices or $S_0 \neq \emptyset$, we know that $H$ is a strict subset of $G$ and by strong connectivity, it must have an out-neighbor in $H^c$. So, by Lemma 33, it must be $b_u$ that has an out-neighbor in $G/H$, say $h$.

Let $S_{u_1}, ..., S_{u_s}$ be the elimination decomposition of $G$. For now, consider deleting $(b_u, w)$ to get $G'$. Note that deleting $(b_u, w)$ might have removed strong connectivity, so in $G'$, there might be vertices in $S_u$ that cannot be reached from $b_u$. Note that all other vertices in $S_u$ can still be reached from $b_u$ in $G'$ since $b_u \in G'$.

So, we can find two subsets $A, B$ such that all edges go from $A$ to $B$ and $w \in A, A \subseteq S_u, b_u \in B$. Now, consider the graph $G'' = G'[B]$ and invoking the characterization theorem with $V_0 = S_0 \cap B, V_i = S_{v_i} \cap B$ for $1 \leq i \leq s$. Note that removing $A$ does not reduce outdegree from $G'$ to $G''$. And from $G$ to $G''$, only the outdegree of $b_u$, which was reduced by 1. So, $V_0$ still is a set of vertices of outdegree $\geq 2$. And since $A$ is a strict subset of $S_u$, we see that we have $V_0, V_1, ..., V_n$ are still strict subsets of $G''$ with $\bigcup_{i=0}^n V_i = V(G'')$.

Now, for property 3 to hold, we only run into a problem if there exists $S_x$ with $b_x = b_u$ that contains all out-neighbors of $b_u$ except $w, h$.

Case 1: Assume such a $S_x$ exists. Then, consider deleting $(b_u, v)$ instead and note that we run into a problem if there exists $S_y$ with $b_y = b_u$ that contains all but $v, h$. However, we see that $b_y \in S_u \cap S_x \cap S_y$, but $v \in S_u \cap S_x \cap S_y^c$ and $w \in S_u \cap S_x^c \cap S_y$, a contradiction to Lemma 57.
Now, consider property 4, and note that it holds unless there exists $S_y$ that contains all out-neighbors of $b_v$ except $v$. Note that $S_w, S_v, S_u$ mutually intersect and therefore, by Lemma 82, they must have a triple intersection. However, this violates Lemma 57 as before. So property 4 must hold.

**Case 2:** Such a $S_x$ does not exist. By symmetry, we may assume there does not exist $S_y$ with $b_y = b_v$ that contains all out-neighbors except $v, h$. So, property 3 holds if we delete $(b_u, w)$ or $(b_u, v)$.

So, consider deleting $(b_u, w)$ again, and note that property 4 holds unless we can find some $S_y$ that contains all out-neighbors of $b_v$ except $w$. Then, delete $(b_u, v)$ and that fails only if $S_z$ contains all out-neighbors of $b_v$ except $v$. But again, $S_u, S_v, S_z$ mutually intersect, and as before, we derive a contradiction.

Therefore, deleting one of $(b_u, v), (b_u, w)$ gives $G'$, which is still of directed pathwidth $\geq 2$, a contradiction.

\[ \square \]

**Lemma 90.** Let $M_u$ be a minimally connected set with more than 4 elimination sets. For any $S_u \subseteq M_u$, there exists $S_w$ such that either $S_u$ hits $S_w$ or $S_w$ hits $S_v$ or $b_v = b_w$. Furthermore, either

1) There exists a non A-type $S_x$ such that $b_v$ has out-neighbors in $S_v \cap S_w^c$ and $S_v^c \cap S_w$ and $N_{out}(b_v) \subseteq S_v \cup S_w$

2) There exists $x \in S_0$ such that such that $x$ has out-neighbors in $S_v \cap S_w^c$ and $S_v^c \cap S_w$ and $N_{out}(x) \subseteq S_v \cup S_w$

3) $b_v = b_u$ and $b_u$ has an out-neighbor in $S_u \cap S_w^c$ and $b_u$ has only 1 out-neighbor in $(S_w \cup S_v)^c$

4) $b_v = b_u$ and $b_u$ has an out-neighbor in $S_u \cap S_w^c$ and $b_u$ has only 1 out-neighbor in $(S_w \cup S_v)^c$

**Proof.** By Lemma 84, we can have at most two minimally connected sets. Furthermore, it is important to note that by Lemma 82, if $S_v$ is any elimination set in $M_u$, then $b_v$ has at most 2 out-neighbors in $M_u^c$.

Now, consider any $S_u \subseteq M_u$ and reducing $v$ to get $S_u'$. By minimality, there must exists an elimination ordering $u_1, ..., u_m$ and its corresponding out-sequence such that $R(G, [u_1, ..., u_m])$ is only 1 vertex. Realize $u_1 \not\in S_0$ or else $N_{out}(u_1) \subseteq S_v$ follows. Also, $u_1 \not\in S_v$, or else it is clear that we cannot eliminate any vertex in $S_v^c$. Thus, $u_1$ is in some elimination set $S_w \neq S_v$. We may assume that $u_1 = w$.

We claim that either $S_v$ hits $S_w$ or $S_u$ hits $S_v$ or $b_v = b_w$. If $S_v \cap S_w \neq \emptyset$, then we are done by Lemma 41 and Lemma 43. So, we may assume that $S_v \cap S_w = \emptyset$. Let $p$ be minimal such that $u_p \not\in S_v$ or $v_p \not\in S_w$.

**Case 1:** $u_p \not\in S_w$. We claim that $u_p \in S_v$. Since $u_p$ can be contracted in $G'$, we see that in $G'$, $N_{out}(u_p) \subseteq S_w$. However, $N_{out}(u_p) \not\subseteq S_w$, so it must be that $u_p$ has $v$ as an out-neighbor. Let $x$ be the out-neighbor of $v$. However, if $u_p \not\in S_v$, then $u_p$ would still have $x$ as an out-neighbor in $G'$ and $x \not\in S_v$ since $x$ is alone. We conclude that $u_p \in S_v$.

Note that since $S_v, S_w$ do not intersect, $p$ is minimal such that $u_p \in S_v$ and since $u_p$ has out-neighbors in $S_v^c$, $u_p = b_v$ is forced. This implies that $N_{out}(u_p) \subseteq S_v \cup S_w$, implies that $S_v$ hits $S_w$.

**Case 2:** $v_p \not\in S_w$. Similarly, we claim that $v_p \in S_v$. Note that outdegree of $u_p$ must have been reduced in $G'$, so $u_p$ must have $x$ as an out-neighbor in $G'$. Since $x \not\in S_w$, it follows that $v_p = x \in S_v$. Then, since $b_w$ has not been reduced, we conclude that $u_p = b_w$. So, $N_{out}(u_p) \subseteq S_v \cup S_w$, implying
that $S_w$ hits $S_v$.

Now, consider the minimal $n$ such that either $u_n \not\in S_v \cup S_w$ or $v_n \not\in S_v \cup S_w$.

**Case 1**: $u_n \not\in S_v \cup S_w$. In $G$, we see that $N_{out}(u_n) \subseteq S_v \cup S_w$ and by Lemma 34, $u_n$ has an out-neighbor in $w_1 \in S_v \cap S^c_w$ and $w_2 \in S^c_v \cap S_w$. We claim that $u_n \in S_0$ or that it is a bad vertex of a non A-type. If not, then either $u_n$ is a bad vertex of an A-type or $u_n$ is contained in some elimination set $S_y$ but is not the bad vertex of $S_y$.

**Case 1.1**: $u_n$ is a bad vertex of an A-type $S_x$ and let $S_x$ hits $S_y$. Since $N_{out}(u_n) \subseteq S_x \cup S_y$, we see that $w_1, w_2 \in S_x \cup S_y$. If $w_1, w_2 \in S_x$, then $w_1 \in S_v \cap S^c_w \cap S_x$ and $w_2 \in S^c_v \cap S_w \cap S_x$, contradicting Lemma 57. Similarly for $S_y$. Without loss of generality, let $w_1 \in S_x \cap S^c_y$ and $w_2 \in S_y \cap S^c_x$. Note that $u_n = b_x \not\in S_v$, but $w_1 \in S_x \cap S_v$, so $S_v$ hits $S_x$ and $v_n \in S_x$. From above, we now split into cases: $S_v$ hits $S^c_w$, $S_w$ hits $S_v$, or $b_v = b_w$.

**Case 1.1.1**: $S_v$ hits $S_w$. Then, $S_w \cap S_x \neq \emptyset$ and since $b_x \not\in S_w$, we conclude that $S_w$ hits $S_x$ and $b_w \in S_x$. Note that $S_y$ cannot hit $S_w$, or else $S_x, S_y, S_w$ hits each other in a cycle. So, $b_w \in S_y$. And, notice $b_w \in S_x \cap S_y \cap S_w$. Since $S_y$ cannot hit $S_x$, it also follows that $b_x \in S_y$. But, $b_x \in S_x \cap S_y \cap S^c_w$ and $w_2 \in S^c_x \cap S_y \cap S_w$, contradicting Lemma 57.

**Case 1.1.2**: $S_y$ hits $S_w$. If $S_y$ hits $S_w$, then we have a cycle of A-types that hit each other in a cycle, so $w_b \in S_y$. Now, either $b_w = b_y$ or $b_w \neq b_y$.

**Case 1.1.2.1**: $b_w = b_y$. Then, since $S_w$ hits $S_v$, $N_{out}(b_y) \subseteq S_w \cup S_v$. Therefore, $H = S_u \cup S_w \cup S_x \cup S_y$ has no out-neighbors to $H^c$, contradicting that $G = H$ has more than 4 elimination sets.

**Case 1.1.2.2**: $b_w \neq b_y$. Then, we know that $b \not\in S_w$ and $S_w$ hits $S_y$. Since $S_w$ hits $S_v, S_y$, we see that $S_x \cap S_y \neq \emptyset$, and $S_y$ cannot hit $S_v$, $b_v \in S_y$. Notice $b_v \in S_x \cap S_y \cap S_v$. Since $S_y$ cannot hit $S_x$, $b_x \in S_y$. But, $b_w \in S_x \cap S_y \cap S^c_w$ and $w_1 \in S_x \cap S^c_v \cap S_w$ contradicts Lemma 57.

**Case 1.1.3**: $b_v = b_w$. Then, since $b_v \in S_x$, we see that $b_v \in S_x \cap S_v \cap S_w$. Note that if $S_y$ hits $S_w$, then $H = S_u \cup S_w \cup S_x \cup S_y$ has no out-neighbors to $H^c$, contradicting that $G = H$ has more than 4 elimination sets. Therefore, $b_w \in S_y$ and $b_w \in S_w \cap S_x \cap S_y$. Since $S_y$ cannot hit $S_x$, it also follows that $b_x \in S_y$. But, $b_x \in S_x \cap S_y \cap S^c_w$ and $w_2 \in S^c_x \cap S_y \cap S_w$, contradicting Lemma 57.

**Case 1.2**: $u_n$ is contained in $S_y$ and $u_n$ is not the bad vertex. Then, $N_{out}(u_n) \subseteq S_y$, but $w_1 \in S_v \cap S^c_w \cap S_y$ and $w_2 \in S^c_v \cap S_w \cap S_y$ contradicts Lemma 57.

We conclude that $u_n \in S_0$ or that it is a bad vertex of a non A-type and they have out-neighbors in $S_v \cap S^c_w$ and $S_w \cap S_v$. Note that $N_{out}(u_n) \subseteq S_v \cup S_w$. Thus, either 1) or 2) holds.

**Case 2**: $v_n \not\in S_v \cup S_w$. Note that since $v_n \not\in S_v \cup S_w$ and $n$ is minimal, we conclude that $u_n = b_v$ or $u_n = b_w$. We claim that $u_n = b_u$. Assume for contradiction that $u_n \neq b_u$, then $u_n$ must be the bad vertex of an A-type in $M_u$. First, we realize that $N_{out}(u_n) \subseteq S_v \cup S_w \cup \{v_n\}$. Now, we split into cases.

**Case 2.1**: $u_n = b_v$. Since $N_{out}(u_n) \subseteq S_v \cup S_w \cup \{v_n\}$, we know that $u_n$ must have an out-neighbor $w_1$ in $S_w \cap S^c_v$. Since $S_v$ is an A-type, it hits $S_x$ and $v_n \in S_x$. Note that if $S_x$ hits $S_w$, then $H = S_u \cup S_v \cup S_w$ has no out-neighbors in $H^c$, contradicting our assumption. Therefore, $b_w \in S_x$.

Also, note that $S_v$ hits $S_w$ is impossible since $b_v$ would have been reduced. So, either $b_v = b_w$ or $S_w$ hits $S_v$.

**Case 2.1.1**: $u_n = b_v = b_w$. In this case, $S_v, S_w$ are both A-types that hit $S_x$. However since
$u_n = b_w$, $u_n$ has at least two out-neighbors in $S_{\mathcal{c}}^w$. Note that if both are in $S_{\mathcal{c}}^w$, then $v_u$ cannot be reduced. So, $u_n$ has an out-neighbor $w_2 \in S_{\mathcal{c}}^w \cap S_v$. But, $b_w \in S_x \cap S_v \cap S_w$ and $u_1 \in S_w \cap S_{\mathcal{c}}^w \cap S_x$ and $w_2 \in S_{\mathcal{c}}^w \cap S_v \cap S_x$, contradicting Lemma 57.

**Case 2.1.2:** $S_v$ hits $S_w$. Note that $S_v$ cannot hit $S_w$, so $b_w \in S_v$. Since $w_1 \in S_x \cap S_w$ and $S_x$ cannot hit $S_w$, we conclude that $b_w \in S_x$. Therefore, $b_w \in S_v \cap S_w \cap S_x$. So, either $S_w$ hits $S_x$ or $b_w = b_x$.

**Case 2.1.2.1:** $S_w$ hits $S_x$. Then, $S_w$ hits $S_x$, $S_v$. So, $S_w$ must have out-neighbors $w_2 \in S_{\mathcal{c}}^w \cap S_x \cap S_v$. But, $w_1 \in S_w \cap S_x \cap S_{\mathcal{c}}^v$, contradicting Lemma 57.

**Case 2.1.2.2:** $b_w = b_x$. Then, $N_{out}(b_x) \subseteq S_v \cup S_w$, so $H = S_v \cup S_w \cup S_x$ has no out-neighbors in $H^c$, contradicting that $G$ has more than 4 elimination sets.

**Case 2.2:** $u_n = b_w$. Note that the same argument for Case 2.1 holds for Case 2.2.

We conclude that $u_n = b_u$ and $u_u$ has at most one out-neighbor outside $(S_u \cup S_w)^c$.

**Theorem 91.** Let $M_u$ be a minimally connected set with more than 4 elimination sets. Then, $M_u$ does not have 2 out-neighbors in $(M_u \cup T_u)^c$.

**Proof.** Assume that $M_u$ has 2 out-neighbors in $(M_u \cup T_u)^c$. By Lemma 83, we have $M_u$, $M_v$ such that $M_u \cup T_u$ and $M_v \cup T_v$ hit each other. Furthermore, $M_v$ has 2 out-neighbors in $(M_u \cup T_u)^c$.

If $M_v$ has 2 out-neighbors in $M_u$ and first assume they are contained in (possibly non-distinct) elimination sets $S_x, S_y \subseteq M_u$ such that either $b_x = b_y$, or $S_x$ hits $S_y$ or $S_y$ hits $S_x$. Let us call this the containment property. Now, if the two out-neighbors of $M_u$ in $M_u^c$ are contained in some elimination set $S_x$, then there exists $a \in M_u$ such that $(u, a) \in E(G)$. Otherwise, $a$ is just an empty vertex. Also, note that $T_u = \emptyset$ since $M_u, M_v$ cannot hit $T_u$ by Lemma 80.

Now, out-contract vertices in $M_u \cap S_u^c \cap S_x \cap \{a, b_u\}^c$. If there are 3 elimination sets in $M_u$, then the previous set was certain non-empty due to the existence of lonely vertices. Now, we appeal to the characterization theorem with $V_0 = S_y$, $V_j = S_{uj} \cap M_u^c$ for $1 \leq j \leq s$, and $V_{s+1} = S_w \cup \{a, b_u\}$ and $V_{s+2} = S_x \cup \{a, b_u\}$. Checking the properties in the characterization theorem, we see that in $G^c$, we still have directed pathwidth $\geq 2$, a contradiction.

So, the containment property does not hold. Then, assume there exists a $S_w \subseteq M_u$ such that $d(S_w, S_w) \geq 2$. By Lemma 90, we can find $S_x$ such that either $S_x$ hits $S_w$ or $S_w$ hits $S_x$ or $b_w = b_x$. Note that $S_x \subseteq M_u$ or else $S_w$ would contain all the out-neighbors of $M_u$ and the containment property holds. And, we claim that there is a vertex $y \in T_u$ that has out-neighbors in $S_x \cap S_{\mathcal{c}}^w$ and $S_x^c \cap S_w$. By Lemma 90, we have the following cases:

**Case 1:** There exists a non A-type $S_y$ such that $b_y$ has out-neighbors in $S_x \cap S_{\mathcal{c}}^w$ and $S_x^c \cap S_w$ and $N_{out}(b_y) \subseteq S_x \cup S_w$. It suffices to show that $S_y$ shares a bad vertex with $S_v$. Since there are only minimally connected sets, we see that $S_y$ must share a bad vertex with $S_u$ or $S_v$. Assume that the former holds.

Let $H$ be the set of elimination sets that have bad vertex $b_u$. Since $M_u$ already has 2 out-neighbors in $(M_u \cup T_u)^c$, by Lemma 80, $M_u$ has no out-neighbor in $H^c \cap M_u$. Therefore, if the former does hold, then $b_u$ has an out-neighbor $w_1 \in S_{\mathcal{c}}^c \cap S_w$ and $w_1 \in H$ is forced. But, $H$ only contain sets of distance 0 and by Lemma 75, $S_w$ cannot have distance $\geq 2$. Then, $M_u$ has both out-neighbors in $S_x, S_w$, and so the containment property holds.
Case 2: There exists \( y \in S_0 \) such that such that \( y \) has out-neighbors in \( S_x \cap S'_w \) and \( S_z \cap S_w \) and \( N_{out}(y) \subseteq S_x \cup S_w \). Since \( S_x \subseteq M_u \), we conclude that \( y \in T_u \).

Case 3: \( b_x = b_u \) and \( b_u \) has an out-neighbor in \( S_w \cap S'_w \) and \( b_u \) has only 1 out-neighbor in \( (S_w \cup S_x)^c \). Since \( b_u \) has an out-neighbor in \( S_w \), which is impossible by similar argumentation in Case 1.

Case 4: \( b_w = b_u \) and \( b_u \) has an out-neighbor in \( S_x \cap S'_w \) and \( b_u \) has only 1 out-neighbor in \( (S_w \cup S_x)^c \). This is not possible because if \( b_w = b_u \), then \( S_w \) has distance 0, a contradiction.

So, our claim holds and \( M_u \) must have an out-neighbor in \( T_u \). If \( M_u \) has 1 out-neighbor in \( M_u \), then note that the out-neighbor must be in \( S_x \cup S_w \). Otherwise, we can reduce \( T_u \) and we would still have a graph of directed pathwidth \( \geq 2 \), by the characterization theorem with \( V_0 = S_0 \cap T_u \) and \( V_j = S_{v_j} \).

However, we can follow the same procedure as described above to contract \( M_u \cap S_w \cap S_x \cap \{a, b_u\} \) and derive a contradiction.

If all \( S_w \subseteq M_u \) has \( d(S_w, S_u) \leq 1 \), then let \( S_x, S_y \) be the two elimination sets that contain all out-neighbors of \( M_u \) and \( T_u \). Then, let \( S_x \) hit \( w_1 \) or \( S_x = w_1 \), where \( b_{w_1} = b_u \). And similarly for \( S_y \), define a \( S_w \). Then out-contract any vertex in \( M_u \cap S_x \cap S_y \cap S_{w_1} \cap S_{w_2} \cap \{a, b_u\} \) and derive a contradiction.

**Theorem 92.** Let \( M_u \) be a minimally connected set with more than 4 elimination sets. Then, \( M_u \) does not have any out-neighbors in \( (M_u \cup T_u)^c \).

**Proof.** Assume that \( M_u \) has out-neighbors in \( (M_u \cup T_u)^c \). By Theorem 91, we know that \( M_u \) has at most 1 out-neighbor in \( (M_u \cup T_u)^c \). By Lemma 84, we know that there is at most one other minimally connected set in \( G \).

Case 1: There exists \( M_v, T_v \) such that \( M_u \cup T_u \) and \( M_v \cup T_v \) hits each other and \( M_v \cup T_v \) also has 1 out-neighbor in \( (M_u \cup T_u)^c \). If \( M_u \) has at least 2 out-neighbors in \( M_u \), then we can follow the argument in Lemma 91 to deduce our result.

Otherwise \( M_u \) only has 1 out-neighbor in \( M_u^c \). This means that \( T_u = \emptyset \). Even if \( M_v \) hits \( T_u \), then we can still contract \( T_u \) and it would still have directed pathwidth \( \geq 2 \).

Now, assume there exists a set \( S_u \subseteq M_u \) such that \( d(S_w, S_u) \geq 1 \). Then by Lemma 90, we can find \( S_x \) such that \( S_w \) hits \( S_x \) or \( S_x \) hits \( S_w \) or \( b_x = b_w \). Furthermore, we have the following cases

**Case 1.1: If there exists a non-A-type \( S_y \) such that \( b_y \) has out-neighbors \( w_1 \in S_w \cap S_x \) and \( w_2 \in S_x \cap S'_w \). Note that \( b_y \neq b_u \) since it has only 1 out-neighbor in \( (M_u \cup T_u)^c \). Therefore, \( b_y = b_u \). Let \( H \) be all set of all distance 0 elimination sets in \( M_u \). Note that if \( w_1, w_2 \notin H \), then Lemma 80 is contradicted. So, one of \( w_1, w_2 \in H \) and we note that \( d(S_w, S_u) \leq 2 \) or else \( S_w, S_x \) would have distance \( \geq 2 \), contradicting Lemma 75.

If exactly one of \( w_1, w_2 \notin H \) then WLOG, let it be \( w_1 \). If \( d(S_w, S_u) = 2 \), then we can find \( S_y, S_z \) such that \( S_w \) hits \( S_y \), which hits \( S_z \), where \( b_z = b_u \). We can contract all else in \( M_u \cap S'_w \cap S_y \cap S'_z \) and show by the characterization theorem that directed pathwidth \( \geq 2 \). So, there are at most 3 elimination sets in \( M_u \). We use a similar argument if \( d(S_w, S_u) = 1 \).

Otherwise, we have \( w_1, w_2 \in H \), there exists \( S_z \) of distance 0 such that \( w_1 \in S_z \). By the same reasoning, we can find \( S_u \) of distance 0 such that \( w_2 \in S_u \). By Lemma 89, \( w_1 \in S_z \cap S'_w \) and \( w_2 \in S_u \cap S'_y \). Now, we may contract all else in \( M_u \cap S'_w \cap S'_y \) and show by equivalence that directed pathwidth \( \geq 2 \). So, there are at most two elimination sets in \( M_u \).
Case 1.2: There exists $y \in S_0$ with out-neighbors in $S_w \cup S_x$. However, note that $S_x \subseteq M_u$ since $M_v$ only has 1 out-neighbors in $(M_v \cup T_v)^c$. So, $y \in T_u$ but $T_u = \emptyset$, contradiction.

Case 1.3: $b_w = b_u$ and and $b_u$ has an out-neighbor in $S_x \cap S_w$. This contradicts the distance assumption of $S_w$.

Case 1.4: $b_x = b_u$ and and $b_u$ has 1 out-neighbor in $(S_w \cup S_x)^c$. has an out-neighbor in $w_1 \in S_w \cap S_x^c$. If $w_1 \notin H$, then follow the argument in Case 1.1 to deduce that there are at most 2 elimination sets. Otherwise, $w_1 \in H$. So, we can find $S_y$ that contains $w_1$ such that $b_y = b_u$. Note that $b_u$ contains two out-neighbors in $S_y^c$ and it already contains an out-neighbor in $M_v \cup T_v$. But, $b_u$ has only 1 out-neighbor in $(S_w \cup S_x)^c$, so $b_u$ has an out-neighbor $w_2 \in S_x \cap S_y^c$. We conclude as in Case 1.1.

If we cannot find a $S_w$ with distance $\geq 1$, we see that all elimination sets are of distance 0. Since $M_u$ only has 1 out-neighbor in $M_u^c$, there must exist $S_x$ of distance 0 such that $b_u$ has an out-neighbor in $S_x$. However, since $b_u$ must have at least 2 out-neighbors in $S_x^c$, we must be able to find $S_y$ of distance 0 such that $b_u$ has an out-neighbor in $S_y$. By Lemma 89, these out-neighbors are in $S_x \cap S_y^c$ and $S_x^c \cap S_y$. Then, we contract all else in $M_u \cap S_x^c \cap S_y$ and deduce that there are at most two elimination sets in $M_u$.

Case 2: There does not exist $M(v, T_v)$. Therefore, $(M_u \cup T_u)^c \subseteq S_0$. However, if $v \in S_0 \cap T_u$, then $v$ has at most 1 out-neighbor in $M_u \cup T_u$ since if $v$ has both out-neighbors in $M_u \cup T_u$, then by 85, $v \in T_u$ must follow. Thus, we may conclude by similar argument in Case 1.

Theorem 93. $M_u$ has at most 4 elimination sets.

Proof. Assume that $M_u$ has more than 4 elimination sets. Then, by Theorem 91, $M_u$ has no out-neighbors outside $M_u \cup T_u$. So, $G = M_u \cup T_u$.

Let $H$ denote the set of all elimination sets of distance 0 in $M_u$. If $M_u$ has out-neighbors in $M_u^c$, then by Lemma 87, there exists only one $v \in T_u$ such that $v$ has out-neighbors in $M_u$. Let $S_x, S_y$ be the elimination sets that contain the 2 out-neighbors of $v$. If one of $S_x, S_y$ has distance $\geq 2$, then if $M_u$ has an out-neighbor $w \in M_u \cap H^c$, then $w \in S_x \cup S_y$, or else we may just out-contract all vertices in $T_u$ into the elimination set of distance $\geq 2$. So, we can follow the cases in Lemma 91.

So, we may assume that $M_u$ has no out-neighbors in $M_u^c$ and $T_u = \emptyset$. By Lemma 80, $H$ has at most 2 out-neighbors in $H^c$. First note that for any $v \in S_v$ of distance $\geq 2$ in $M_u$, by Lemma 90, there must exist $S_w$ such that $b_u$ has out-neighbors in $S_w \cap S_v^c$ and $S_v \cap S_w^c$.

We claim that $M_u$ has no elimination sets of distance $\geq 4$. Assume otherwise and note that we can find $S_v \subseteq M_u$ be an elimination set in $M_u$ such that $S_v$ is not hit by another elimination set and its distance is $\geq 4$. Then, by Lemma 90, there must exist $S_w$ such that either $S_v$ hits $S_w$ or $b_w = b_v$ and $b_u$ has out-neighbors in $w_1 \in S_w \cap S_v^c$ and $w_2 \in S_v^c \cap S_v$ and $N_{out}(b_u) \subseteq S_v \cup S_w$. Note that $S_v$ has distance $\geq 3$. Note that $S_w$ hits another elimination set $S_x$, which has distance $\geq 2$, so there exist $S_y$ such that $b_u$ has out-neighbors in $w_3 \in S_y^c \cap S_x$ and $w_4 \in S_x \cap S_y$. Note that $w_3 \neq w_4$ and $w_1 \neq w_2$. Since $b_u$ has at most 2 out-neighbors in $H^c$ and $w_1, w_2, w_3 \in H^c$ by Lemma 75, we must have $w_2 = w_3$ and $w_4 \in H$. But $w_4 \in H$ contradicts $N_{out}(b_u) \subseteq S_v \cup S_w$.

If $S_v \subseteq M_u$ is an elimination set of distance $\geq 2$, then the same argument above shows that there must exists $S_w$ such that $N_{out}(b_u) \subseteq S_v \cup S_w$. If $S_w$ hits $S_v$, then since $S_v$ is of distance 2, there exists $S_{u_1}, S_{u_2}$ such that $S_v$ hits $S_{u_1}$ and $S_{u_1}$ hits $S_{u_2}$ with $b_{u_2} = b_u$. By strong connectivity, note that $S_w, S_v, S_{u_1}, S_{u_2}$ contains all its out-neighbors. So, we conclude that there are at most 4 elimination sets.
Proof. We know that there exists a simple directed path from all bad vertices are still in the strongly connected component. Let claim that this path is bounded. Let ordering (and disregarding in-contractions), let it be ordering. Assume we can find such that there exists sets are of distance \(\leq 1\). If there exists \(S_v \subseteq M_u\) a elimination set of distance 1, then by 90, we have \(S_w\) with the given properties and the following cases:

**Case 1:** Note that \(b_u\) has out-neighbors in \(S_u \cap S_w\) and \(S_v^c \cap S_w\) and \(N_{out}(b_u) \subseteq S_v \cup S_w\). Let \(S_v \) hit \(S_{u_1}\) and \(b_{u_1} = b_u\), then clearly, \(S_v, S_w, S_{u_1}\) contains all its out-neighbors and we conclude there are at most 4 elimination sets

**Case 2:** This is impossible as \(T_u = \emptyset\)

**Case 3:** \(b_u = b_w\) is impossible as the distance of \(S_v\) is 1.

**Case 4:** \(b_w = b_u\) and \(b_u\) has an out-neighbor in \(S_v \cap S_w^c\) and \(b_u\) has 1 out-neighbor in \(w_1 \in (S_w \cup S_v)^c\). Let \(S_{u_1}\) contain \(w_1\) and since it is of distance \(\leq 1\), there exists \(S_{u_2}\), where \(S_{u_1} = S_{u_2}\) or \(S_{u_1}\) hits \(S_{u_2}\) and \(b_{u_2} = b_u\). Then, we conclude since \(S_v, S_w, S_{u_1}, S_{u_2}\) contains all of its out-neighbors.

So we may assume that \(G = H\). Assume we can find \(S_v, S_w\) such that there exists out-neighbors of \(b_u, u_1, u_2\), such that \(u_1 \in S_v^c \cap S_w\) and \(u_2 \in S_{v_1} \cap S_{v_2}\). If \(G = S_{v_1} \cup S_{v_2}\), then it at most 4 elimination sets. Otherwise, note that there must be an out-neighbor of \(b_u\) in \((S_v \cup S_w)^c\) and say \(S_3 \in S_v\). Note that if both \(u_1, u_2 \in S_x\), then since \(b_u \in S_v \cup S_w \cup S_x\), we contradict Lemma 57. Without loss of generality, let \(u_1 \in S_v^c \cap S_w \cap S_{x_1}\). By Lemma 88, we have at most 3 elimination sets since \(u_3 \in S_v^c \cap S_{w} \cap S_x\).

If we cannot find such \(S_v, S_w\), then we claim that we can find one elimination set \(S_x\) that contain \(N_{out}(b_u)\). Assume otherwise, then we can find \(S_y\) such that \(|S_y \cap N_{out}(b_u)|\) is maximized. By our assumption, we can find \(u_1 \in N_{out}(b_u)\) such that \(u_1 \notin S_y\). However, we can also find \(S_z\) such that \(u_1 \in S_z\). Note that if there exists \(u_1 \in S_y \cap S_z^c\), then we are done. So, we conclude that \(|S_y \cap N_{out}(b_u)| < |S_z \cap N_{out}(b_u)|\), contradicting our assumption. Therefore, \(S_x\) contains all the out-neighbors of \(b_u\) and we have only 1 elimination set in \(G\).

\[\square\]

### 8.3 Finiteness of Forbidden Minors

**Theorem 94.** Let \(S_u\) be an elimination set, then \(|S_u| \leq 160000\).

*Proof.* We know that there exists a simple directed path from \(u\) to \(b_u\) by following the elimination ordering (and disregarding in-contractions), let it be \(P_{ab_u}\). We call this the main path. We first claim that this path is bounded. Let \(v_1, ..., v_k\) be vertices on this path, where \(v_1 = u, v_k = b_u\). Then, for \(1 \leq i < k\), consider the graph obtained by out-contracting \((v_i, v_{i+1})\), say \(G'\). Note that if \(G'\) is not strongly connected, then we can consider strongly connected components with as long as all bad vertices are still in the strongly connected component.

We want to appeal to the characterization theorem with \(V_0 = S_0, V_j = S_{w_j} \cap G'\) for \(1 \leq j \leq s\). The first few properties hold trivially. Note that \(S_0\) has vertices of outdegree \(\geq 2\) since if \(x \in S_0\) has lost outdegree, then \(N_{out}(x) \subseteq S_u\), which would imply \(x \in S_v\), a contradiction. So, by the characterization theorem, there must exists \(V_j = S_{v_j}\) such that either property 3 fails or property 4 fails.

If property 3 fails, then we see that either \(b_{w_j}\) had \(v_i, v_{i+1} \in V_j^c\) as out-neighbors, \(b_{w_j} = v_i\) or \(b_{w_j} = v_{i+1}\) and it has \(v_i \in V_j^c\) as an out-neighbor. If property 4 fails, then it must be that \(v_i \in V_j^c, v_{i+1} \in V_j\). But, note that the number of elimination sets and bad vertices are bounded to 8. Note that by if \(v_i \in V_j^c, v_{i+1} \in V_j\), then \(v_k \in V_j\) unless there exists \(l\) such that \(v_l = b_{w_j}\). Therefore, we conclude that for each bad vertex \(b_{w_j}\) can only prevent the contraction of at most 16 vertices. Furthermore, the second condition can only prevent the contraction of at most 8 vertices. We also have to make sure that we do not disconnect an elimination set from its bad vertex. It suffices to
show that for any elimination set \( S_v \), \( P_{xb_v} \) is still a directed path. This is only a problem when \( v_i \) is an intersection vertex for \( P_{xb_v} \) and \( P_{ab_u} \) and \( v_i+1 \) is not, since if both \( v_i, v_i+1 \) are intersection points, then we may either cycle contract or out-contract without problems. However, consider contracting \( v_i+1 \) and note that we run into a problem only if property 3 fails and \( v_i+1, v_i+2 \) is a bad vertex, so there are at most 8 more possible vertices when this can be a problem. Since there are 8 different paths \( P_{xb_v} \), gives an additional 8*8 = 64 vertices. Therefore, \( P_{ab_u} \) has at most 100 vertices.

Consider the maximal elimination ordering of \( S_u \), say \( u_1, ..., u_m \) and it outsequence \( w_1, ..., w_m \). If there exists a vertex \( x \) such that \( w_i, ..., w_{i+k} = x \), then we claim that \( k \leq 25 \). Now, consider \( u_{i+1}, ..., u_{i+k} \) and find a \( u_{i+l} \) such that it does not have \( u_{i+1}, ..., u_{i+k} \) as its in-neighbor. Consider contracting directed path from \( u_{i+l} \) to \( x \) in \( G \). Note that property 3 holds, we can find \( V_j = S_{v_j} \) such that unless a bad vertex is on that path or that the bad vertex is \( w \) and one of its two out-neighbors in \( V^c \) is on that path. Note that property 4 holds since \( u_{i+1} \), ..., \( u_{i+k} \) as its in-neighbor. So, there are at most 16 such paths. By the same argument before, there are at most 100 vertices on these paths.

So, our most conservative bound is that for each 100 vertices on \( P_{ab_u} \), there could be these 16 paths that are reduced at each vertex along \( P_{ab_u} \), each of which has 100 vertices. Hence \( S_u \) has at most 100*16*100 = 160000 vertices.

**Theorem 95.** Let \( G \) be a forbidden minor, then \(|S_0| \leq 7\)

**Proof.** Note that if there exists two minimally connected sets, \( M_u, M_v \), then by the argument in Lemma 83, Lemma 84, we see that \(|S_0 \cap T^c_u \cap T^c_v| \leq 1\). Since \(|T_u|, |T_v| \leq 3\) by Lemma 87, we conclude that \(|S_0| \leq 7\).

Otherwise, note that if no minimally connected sets exist, then we must have A-types that hit each other in a cycle. This implies that \( S_0 = \emptyset \). This leaves us the case when we only have one minimally connected set \( M_u \). Note that if \((M_u \cup T_u)^c = \emptyset\), then \( S_0 = T_u \), so \(|S_0| \leq 3\).

Otherwise, \( M_u \cup T_u \) only has 1 out-neighbor \( x \in (M_u \cup T_u)^c = S_0 \cap T^c_u \). Let us consider the case when \((b_u, x) \notin E(G) \) and \((b_u, x) \in E(G) \). Now, by Lemma 86, note that if we can find \( u, v \in S_0 \cap T^c_u \), such that out-contracting \((u, v)\) still preserves the outdegree of the vertices in \( S_0' \cap T^c_u \), then we derive a contradiction. Since \(|T_u| \leq 3\), we conclude that \(|S_0| \leq 7\).

**Theorem 96.** \( \mathcal{F}' \) is finite.

**Proof.** It suffices to show that \( \mathcal{F}' \) only contains graphs with a bounded vertex count. Let \( G \subseteq \mathcal{F}' \), By Lemma 84 and Theorem 93, we conclude that there are at most 8 elimination sets in \( G \). So, by Theorem 94 and Theorem 95, we conclude that \(|V(G)| \leq |S_0| + \sum_{i=1}^8 |S_{v_i}| \) is bounded.

**References**


