Forbidden Directed Minors and Directed Pathwidth

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Abstract

Undirected graphs of pathwidth at most one are characterized by two forbidden minors i.e., (i) K_3 the complete graph on three vertices and (ii) $S_{2,2,2}$ the spider graph with three legs of length two each [BFKL87]. Directed pathwidth is a natural generalization of pathwidth to digraphs. In this paper, we prove that digraphs of directed pathwidth at most one are characterized by a finite number of forbidden *directed minors*. To achieve our goal, we present a new decomposition theorem for digraphs of directed pathwidth ≥ 2 and prove several properties of the forbidden directed minors, which are of independent interest.

Keywords: directed pathwidth, forbidden minors, graph minors, pathwidth.

1 Introduction

Pathwidth measures how close an undirected graph is to being a path. Directed pathwidth, introduced by Reed, Seymour and Thomas (see [Bar06]), measures how close a digraph is to being a directed path. For an undirected graph (resp. digraph) G, let $\mathsf{pw}(G)$ (resp. $\mathsf{dpw}(G)$) denote its pathwidth (resp. directed pathwidth). Directed pathwidth is a generalization of pathwidth to digraphs i.e., for an *undirected* graph G, let $\overleftrightarrow{G}$ be the *digraph* obtained by replacing each edge $\{u, v\}$ of G by two directed edges (u, v) and (v, u), then, $\mathsf{dpw}(\overleftrightarrow{G}) = \mathsf{pw}(G)$ [Bar06, Lemma 1].

Pathwidth has several equivalent characterizations in terms of vertex separation number, node searching number, interval thickness (i.e., one less than the maximum clique size in an interval supergraph) and cops and (invisible and eager) robber games. Similarly, directed pathwidth is characterized by directed vertex separation number and cops and (invisible and eager) robber games on digraphs [YC08]. Both pathwidth and directed pathwidth have associated decomposition structures called path decomposition and directed path decomposition respectively.

The graph minor theorem [RS04] (i.e., undirected graphs are well-quasi-ordered under the minor relation) implies that every minor-closed family of undirected graphs has a finite set of minimal forbidden minors. In particular, it implies that for all $k \geq 0$, graphs of treewidth (or pathwidth) $\leq k$ are characterized by a finite set of forbidden minors. The complete sets of forbidden minors are known for small values of treewidth and pathwidth. A graph has treewidth at most one (resp. two) if and only if it is K_3-free (resp. K_4-free). Graphs of treewidth at most three are characterized by four forbidden minors (K_5, the graph of the octahedron, the graph of the pentagonal prism,

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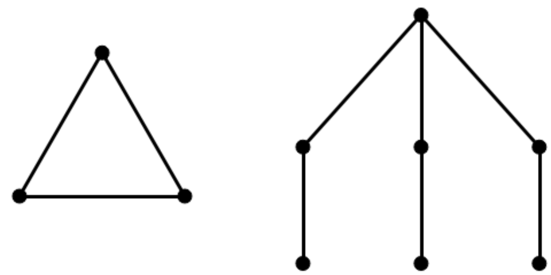

Figure 1: The forbidden minors of undirected graphs of pathwidth at most one: K_3 and $S_{2,2,2}$

and the Wagner graph) [APC90, ST90]. Graphs of pathwidth at most one are characterized by two forbidden minors [BFKL87] i.e., (i) K_3 the complete graph on three vertices and (ii) $S_{2,2,2}$ the spider graph with three legs of length two each (see Figure 1). Graphs of pathwidth at most two are characterized by 110 forbidden minors [KL94].

A natural question is "*are digraphs of bounded directed pathwidth characterized by a finite set of forbidden directed minors* ?". Unfortunately there is no generalization of the graph minor theorem for digraphs yet. Existing notions of directed minors (eg. directed topological minors [Hun07], butterfly minors [JRST01], strong contractions [KS12], directed immersions [CS11]) do not imply well-quasi-ordering of all digraphs (see [Kin13] for more details).

Recently, the first author [Kin13] introduced a notion of *directed minors* based on several operations of contracting special subsets of directed edges, conjectured that digraphs are well-quasi-ordered under the proposed *directed minor relation* and proved the conjecture for some special classes of digraphs. This conjecture implies that every family of digraphs closed under the directed minor operations (see Section 2) are characterized by a finite set of minimal *forbidden directed minors*. In particular, it implies that for all $k \geq 0$, digraphs of Kelly-width (or DAG-width, or directed pathwidth) $\leq k$ are characterized by a finite set of forbidden directed minors (see [Kin13] for more details).

Using this directed minor relation, the authors studied the *forbidden directed minors* for digraphs with Kelly-width one and two (i.e., partial 0-DAGs and partial 1-DAGs) and proved that partial 0-DAGs (simply called DAGs) are characterized by one forbidden directed minor (i.e., K_2) and partial 1-DAGs are characterized by three forbidden directed minors (see [KZ13] for more details). Kelly-width is a generalization of treewidth to digraphs.

A digraph has directed pathwidth zero if and only if it is a DAG and hence it is characterized by one minimal forbidden minor i.e., K_2 (see [KZ13]). In this paper, we show the finiteness of the set of minimal forbidden directed minors of digraphs with directed pathwidth one.

1.1 Notation and Preliminaries

We use standard graph theory notation and terminology (see [Die05]). For a directed graph (digraph) G, we write $V(G)$ for its vertex set (or node set) and $E(G)$ for its edge set. All digraphs in this paper are finite and simple (i.e., no self loops and no multiple edges) unless otherwise stated. For $X \subseteq V(G)$, let $|X|$ denote the number of vertices in X. For an edge $e = (u, v)$, we say that e is an edge *from* u *to* v. We say that u is the *tail* of e and v is the *head* of e. We also say that u is an in-neighbor of v and v is an out-neighbor of u. The out-neighbors of a vertex u is given by

$N_{out}(u) = \{v : (u,v) \in E\}$ and the in-neighbors of a vertex v is given by $N_{in}(v) = \{u : (u,v) \in E\}$. Let $d_{out}(u) = |N_{out}(u)|$ (resp. $d_{in}(v) = |N_{in}(v)|$) denote the out-degree of u (resp. in-degree of v).

For $S \subseteq V(G)$ we write $G[S]$ for the subgraph induced by S, and $G \setminus S$ for the subgraph induced by $V(G) \setminus S$. For $F \subseteq E(G)$, we write $G[F]$ for the subgraph with vertex set equal to the set of endpoints of F, and edge set equal to F.

For a digraph G, let $\overline{G}$ be the undirected graph, where $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{\{u,v\} : (u,v) \in E(G)\}$. We say that $\overline{G}$ is the *underlying* undirected graph of G. For an *undirected* graph G, let $\overleftrightarrow{G}$ be the *digraph* obtained by replacing each edge $\{u,v\}$ of G by two directed edges (u,v) and (v,u). We say that $\overleftrightarrow{G}$ is the *bidirected* graph of G.

A *directed (simple) path* in G is a sequence of vertices $v_1, v_2, \ldots, v_l$ such that for all $1 \leq i \leq l-1$, $(v_i, v_{i+1}) \in E(G)$. For a subset $X \subseteq V(G)$, the set of vertices reachable from X is defined as: $Reach_G(X) := \{v \in V(G) :$ there is a directed path to v from some $u \in X\}$. We say that G is *weakly connected* if $\overline{G}$ is connected. We say that G is *strongly connected* if, for every pair of vertices $u, v \in V(G)$, there is a directed path from u to v and a directed path from v to u.

We use the term DAG when referring to directed acyclic graphs. Let T be a DAG. For two distinct nodes i and j of T, we write $i \prec_T j$ if there is a directed walk in T with first node i and last node j. For convenience, we write $i \prec j$ whenever T is clear from the context. For nodes i and j of T, we write $i \preceq j$ if either $i = j$ or $i \prec j$. For an edge $e = (i,j)$ and a node k of T, we write $e \prec k$ if either $j = k$ or $j \prec k$. We write $e \sim i$ (resp. $e \sim j$) to mean that e is incident with i (resp. j).

Let $\mathcal{W} = (W_i)_{i \in V(T)}$ be a family of finite sets called *node bags*, which associates each node i of T to a node bag W_i. We write $W_{\succeq i}$ to denote $\bigcup_{j \succeq i} W_j$.

The following notion of *guarding* is useful :

Definition 1. [Guarding] Let G be a digraph and $W, X \subseteq V(G)$. We say X *guards* W if $W \cap X = \emptyset$, and for all $(u,v) \in E(G)$, if $u \in W$ then $v \in W \cup X$.

In other words, X guards W means that there is no directed path in $G \setminus X$ that starts from W and leaves W.

Definition 2. [Edge contraction] Let G be an undirected graph, and $e = \{u,v\} \in E(G)$. The vertices and edges of the graph G' obtained from G by *contracting* e are:

- $V(G') = V(G) \setminus u$
- $E(G') = (E(G) \cup \{\{x,v\} : \{x,u\} \in E(G)\}) \ \setminus \ \{\{x,u\} : x \in V(G)\}$

Definition 3. [Minor] Let G and H be undirected graphs. We say that H is a *minor* of G, (denoted by $H \leq G$), if H is isomorphic to a graph obtained from G by a sequence of vertex deletions, edge deletions and edge contractions. These operations may be applied to G in any order, to obtain its directed minor H.

Theorem 4. (Robertson-Seymour theorem [RS04]) Undirected graphs are well-quasi-ordered by the minor relation $\leq$.

Corollary 5. Every minor-closed family of undirected graphs has a finite set of minimal forbidden minors. In particular, for all $k \geq 0$, graphs of treewidth (or pathwidth) $\leq k$ are characterized by a finite set of forbidden minors.

2 Directed Minors

In this section, we present a subset of the *directed minor operations* from [Kin13]. The *directed minor relation* in [Kin13] has more operations that are not necessary for our results in the current paper.

When we perform the following operations on a digraph G to obtain a digraph G', we remove any resulting self-loops and multi-edges from G'. Cycle contraction is a generalization of the edge contraction from Definition 2.

Definition 6. [Cycle contraction] Let G be a graph, and $C = \{v_1, v_2, \ldots, v_l\} \subseteq V(G)$ be a directed cycle in G i.e., $(v_i, v_{i+1}) \in E(G)$ for $1 \le i \le l-1$ and $(v_l, v_1) \in E(G)$. The vertices and edges of the graph G' obtained from G by *contracting* C are:

- $V(G') = \{V(G) \setminus C\} \cup \{w\}$, where w is a new vertex.
- $E(G') = \{E(G) \setminus \{\{(x, v_i) : 1 \le i \le l, x \in V(G)\} \cup \{(v_i, x) : 1 \le i \le l, x \in V(G)\}\}\}$ $\cup \; \{\{(x, w) : (x, v_i) \in E(G)$ for some $x \in V(G) \setminus C, 1 \le i \le l\} \cup \{(w, x) : (v_i, x) \in E(G)$ for some $x \in V(G) \setminus C, 1 \le i \le l\}\}$

Butterfly contractions (defined by Johnson et al [JRST01]) allow us to contract a directed edge $e = (u, v)$ if either e is the only edge with head v, or it is the only edge with tail u, or both. The following operations, *out-contraction* and *in-contraction*, are slightly general and allows *any* edge to be out-contracted or in-contracted after removing certain incident edges. Out-contracting an edge (u, v) is equivalent to removing all the out-going edges of u and identifying u and v. Similarly, in-contracting an edge (u, v) is equivalent to removing all the in-coming edges of v and identifying u and v. Hence, we can out-contract or in-contract *any* edge of G without creating new paths in G.

Definition 7. [Out contraction] Let G be a graph, and $e = (u, v) \in E(G)$. The vertices and edges of the graph G' obtained from G by *out-contracting* e are:

- $V(G') = V(G) \setminus u$
- $E(G') = (E(G) \cup \{(x, v) : (x, u) \in E(G)\}) \ \setminus \ \{(x, u), (u, x) : x \in V(G)\}$

Note that we delete the vertex u, the tail of the edge $e = (u, v)$. The vertex v exists in both G and G'.

Definition 8. [In contraction] Let G be a graph, and $e = (u, v) \in E(G)$. The vertices and edges of the graph G' obtained from G by *in-contracting* e are:

- $V(G') = V(G) \setminus v$
- $E(G') = (E(G) \cup \{(u, x) : (v, x) \in E(G)\}) \ \setminus \ \{(x, v), (v, x) : x \in V(G)\}$

Note that we delete the vertex v, the head of the edge $e = (u, v)$. The vertex u exists in both G and G'.

Definition 9. [Directed minor] Let G and H be digraphs. We say that H is a *directed minor* of G, (denoted by $H \precsim G$), if H is isomorphic to a graph obtained from G by a sequence of vertex deletions, edge deletions, cycle contractions, out/in contractions. These operations may be applied to G in any order, to obtain its directed minor H.

3 Directed Pathwidth

3.1 Definition

Definition 10. [Directed path decomposition and Directed pathwidth [Bar06]] A *Directed path decomposition* of a digraph G is a sequence $X_1, X_2, \ldots, X_r$ of subsets (node bags) of $V(G)$, such that:

- $\bigcup_{1 \leq i \leq r} X_i = V(G)$. (DPW-1)
- For all $i, j, k \in [r]$, if $i \leq j \leq k$, then $X_i \cap X_k \subseteq X_j$. (DPW-2)
- For all arcs $(u, v) \in E(G)$, there exist i, j with $1 \leq i \leq j \leq r$ such that $u \in X_i$ and $v \in X_j$. (DPW-3)

The width of a directed path decomposition $\mathcal{X} = (X_i)_{i \in [r]}$ is defined as $\max\{|X_i| : 1 \leq i \leq r\} - 1$. The *directed pathwidth* of G, denoted by $\mathsf{dpw}(G)$, is the minimum width over all possible directed path decompositions of G.

(DPW-2) can be replaced by the following equivalent statement:

- For any $v \in V(G)$, $\{i : X_i \cap v \neq \emptyset, 1 \leq i \leq r\}$ is an integer interval. (DPW-2')

(DPW-3) can be replaced by the following equivalent statement:

- For any i with $1 < i < r$, there is no edge *from* $\bigcup_{i+1 \leq j \leq r} X_j$ *to* $\bigcup_{1 \leq j \leq i-1} X_j$ in $G \setminus X_i$. (DPW-3')

A directed path decomposition is called *incremental* if it satisfies the following condition:

- For any $i \in [r-1]$, $|X_i \Delta X_{i+1}| = 1$. (DPW-4)

3.2 Basic properties

A digraph D is a *directed union* of digraphs D_1 and D_2 if D_1 and D_2 are induced subgraphs of D, $V(D_1) \cup V(D_2) = V(D)$, and no edge of D has head in $V(D_1)$ and tail in $V(D_2)$. Directed pathwidth is closed under directed unions (see [YC08]). The following theorem is immediate.

Theorem 11. ([YC08]) Directed pathwidth of a digraph G is equal to the maximum directed pathwidth taken over the strongly-connected components of G.

Hence, all digraphs in the rest of this paper are strongly connected, unless otherwise stated. Kim and Seymour [KS12, Kim13] proved that directed pathwidth is closed under strong minors and butterfly minors. The following lemma is immediate.

Lemma 12. Let $H \precsim G$. Then, $\mathsf{dpw}(H) \leq \mathsf{dpw}(G)$. In other words, directed pathwidth is monotone under the directed minor operations mentioned in Section 2.

4 Basic Forbidden Minors

Let $\mathcal{F}$ be the set of all minor-minimal (w.r.t $\preceq$) digraphs of directed pathwidth two. Our goal is to prove that $\mathcal{F}$ is finite. By Theorem 11 we may assume that all digraphs in $\mathcal{F}$ are strongly-connected. For a digraph G, let $\tilde{G}$ be the digraph obtained by reversing the directions of all the edges in G.

Lemma 13. ([YC08, Tam11]) For any digraph G, $\mathsf{pw}(G) = \mathsf{pw}(\tilde{G})$.

Lemma 14. For digraphs G and H, $H \preceq G$ if and only if $\tilde{H} \preceq \tilde{G}$.

Theorem 15. (Kintali, Zhang [KZ13]) Let G be a simple digraph such that every vertex in $V(G)$ has *out-degree* ≥ 2. Then G contains K_3, N_4 or M_5 (see Figure 2) as a directed minor.

Theorem 16. Let G be a simple digraph such that every vertex in $V(G)$ has *in-degree* ≥ 2. Then G contains $K_3, \tilde{N}_4$ or $\tilde{M}_5$ as a directed minor.

Corollary 17. $K_3, N_4, M_5, \tilde{N}_4, \tilde{M}_5 \in \mathcal{F}$. We say that these five digraphs are the *"basic forbidden minors"*.

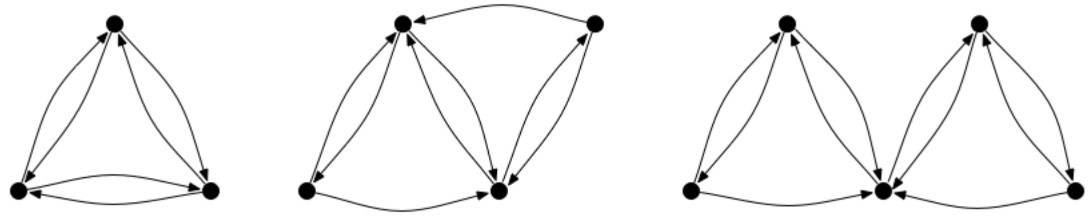

Figure 2: K_3, N_4 and M_5

Let $\mathcal{F}' = \mathcal{F} \setminus \{K_3, N_4, M_5, \tilde{N}_4, \tilde{M}_5\}$. Since $\mathsf{dpw}(S_{2,2,2}) = 2$, (and $S_{2,2,2}$ is a forbidden minor for undirected graphs of pathwidth at most one) we have $S_{2,2,2} \in \mathcal{F}'$. So $\mathcal{F}'$ is non-empty. The rest of this paper is aimed at proving that $\mathcal{F}'$ is finite. Let $G \in \mathcal{F}'$. The following three properties are crucial.

- G is strongly-connected (PROP-1)
- G has at least one vertex of out-degree equal to one (PROP-2)
- G has at least one vertex of in-degree equal to one (PROP-3)

5 A New Decomposition Theorem

In this section, we present a new structural characterization of digraphs with directed pathwidth ≥ 2. For $X \subseteq V(G)$, the complement of X is defined as $X^c := V(G) \setminus X$. For $X, Y \subseteq V(G)$ we say that X *hits* Y if $Y \setminus X$ guards X. X *has an out-neighbor* in Y if there exists $u \in X$ and $v \in Y$ such that $(u, v) \in E(G)$. A vertex u has k out-neighbors in Y if there exist distinct $v_1, .., v_k \in Y$ such that $(u, v_i) \in E(G)$ for all $i \in [k]$.

5.1 Elimination Orderings

Definition 18. Let G be a digraph and $u \in V(G)$ with $d_{out}(u) = 1$, then $R(G, u)$ is defined as the digraph obtained by out-contracting u. We say that $R(G, u)$ is obtained by *eliminating* u or *reducing* u in G. If $d_{out}(u) \neq 1$, then $R(G, u)$ is not defined.

In other words, *eliminating* or *reducing* a vertex u in a digraph G yields digraph $G' := R(G, u)$ that is obtained from G by deleting vertex u and adding arcs from all in-neighbors of u to the unique out-neighbor of u, without introducing loops or multiple edges. In the rest of this paper whenever we use the notation $R(G, u)$, we assume that $d_{out}(u) = 1$ in G.

Observation 19. *If G is strongly-connected then $R(G, u)$ is also strongly-connected.*

Definition 20. We say that $u_1, ..., u_m$ is an **elimination ordering** in G, if $G_0 := G$ and $G_i := R(G_{i-1}, u_i)$ is defined for $i \in [m]$ i.e., $d_{out}(u_i) = 1$ in G_{i-1}. The corresponding **out-sequence** for this elimination ordering is a sequence of vertices $v_1, ..., v_m$, where v_i is the unique out-neighbor of u_i in G_{i-1} for $i \in [m]$. We also denote G_i as $R(G, [u_1, ..., u_i])$.

Observation 21. *Let $u_1, ..., u_m$ be an elimination ordering in G and $v_1, ..., v_m$ be its out-sequence, then $N^G_{out}(u_i) \subseteq \{u_1, ..., u_{i-1}, v_1, ..., v_{i-1}, v_i\}$.*

Definition 22. Let $u_1, ..., u_m$ be an elimination ordering in G and $v_1, ..., v_m$ be its out-sequence. We say that this elimination ordering is **linear** if either $u_{i+1} = v_i$ or $v_{i+1} = v_i$ holds for $1 \leq i < m$.

Observation 23. *Let $u_1, ..., u_m$ be a linear elimination ordering and $v_1, ..., v_m$ be its corresponding out-sequence, then $v_i \in \{u_1, ..., u_m, v_m\}$ for $i \in [m]$.*

Lemma 24. Let $u_1, ..., u_m$ be a linear elimination ordering and $v_1, ..., v_m$ be its corresponding out-sequence, then $\{u_1, ..., u_i\}$ hits $\{v_i\}$ for all $1 \leq i \leq m$

Proof. We proceed with induction on i for $1 \leq i \leq m$. Note that $\{u_1\}$ hits $\{v_1\}$ since u_1 has only one out-neighbor. Assume that $\{u_1, ..., u_{i-1}\}$ hits $\{v_{i-1}\}$. Since u_i has out-degree 1 at time of contraction, we have $N_{out}(u_i) \subseteq \{u_1, ..., u_{i-1}, v_i\}$. By linearity, either $v_{i-1} = u_i$ or $v_{i-1} = v_i$.
Case 1: $v_{i-1} = u_i$. Now, consider any $w \in \{u_1, ..., u_i\}$. If $w \neq u_i$, then by the inductive hypothesis, $N_{out}(w) \subseteq \{u_1, ..., u_{i-1}, u_i\}$. If $w = u_i$, then $N_{out}(w) \subseteq \{u_1, ..., u_{i-1}, v_i\}$. We conclude that $\{u_1, ..., u_i\}$ hits $\{v_i\}$.
Case 2: $v_{i-1} = v_i$. Now, consider any $w \in \{u_1, ..., u_i\}$. If $w \neq u_i$, then by the inductive hypothesis, $N_{out}(w) \subseteq \{u_1, ..., u_{i-1}, v_i\}$. If $w = u_i$, then $N_{out}(w) \subseteq \{u_1, ..., u_{i-1}, v_i\}$. We conclude that $\{u_1, ..., u_i\}$ hits $\{v_i\}$. □

In the rest of this paper, *all elimination orderings are linear.* We say a digraph G *admits* an elimination ordering if there exists an elimination ordering $u_1, ..., u_m$ such that $R(G, [u_1, ..., u_m])$ is a single vertex.

Lemma 25. A strongly connected digraph G has directed pathwidth at most 1 if and only if it admits an elimination ordering. In other words, digraphs in $\mathcal{F}$ do not admit an elimination ordering.

Definition 26. An elimination ordering $u_1, ..., u_m$ in G is **maximal** if there does not exist $u_{m+1} \in V(G)$ such that $u_1, ..., u_{m+1}$ is also an elimination ordering in G.

Observation 27. *Let $G \in \mathcal{F}$. Let $u_1, ..., u_m$ be an elimination ordering in G and $v_1, ..., v_m$ be its corresponding out-sequence. It is maximal if and only if in $R(G, [u_1, ..., u_m])$, v_m has outdegree ≥ 2 and all in-neighbors of v_m have outdegree ≥ 2.*

Lemma 28. *Let $u_1, ..., u_m$ be a maximal elimination ordering and let $u'_1, ..., u'_l$ be an elimination ordering. If $u'_1 \in \{u_1, ..., u_m\}$, then $u'_i \in \{u_1, ..., u_m\}$ for all $1 \leq i \leq l$.*

Proof. Let $u'_1, ..., u'_l$ be an elimination ordering and $v'_1, ..., v'_l$ be its corresponding out-sequence. We use strong induction on i. For the base case of $i = 1$, it holds by assumption. Now, assume that $u'_j \in \{u_1, ..., u_m\}$ for $1 \leq j < i$.

For the sake of contradiction, assume $u'_i \notin \{u_1, ..., u_m\}$. Since u'_i can be contracted in $R(G, [u'_1, ..., u'_{i-1}])$, we see that in G, $N_{out}(u'_i) \subseteq \{u'_1, ..., u'_{i-1}, v'_i\}$. Also, by Lemma 24, we see that $\{u'_1, ..., u'_{i-1}\}$ hits $\{v'_{i-1}\}$. Thus, since $u'_1, ..., u'_{i-1}$ were reduced in the elimination ordering $u_1, ..., u_m$, we see that in $R(G, [u_1, ..., u_m])$, the set $\{u'_1, ..., u'_{i-1}, v'_{i-1}\}$ must have been contracted to at most 1 vertex. Now, by linearity, either $v'_i = v'_{i-1}$ or $u'_i = v'_{i-1}$.

Case 1: $v'_i = v'_{i-1}$. Then, $N_{out}(u'_i) \subseteq \{u'_1, ..., u'_{i-1}, v'_{i-1}\}$. In $R(G, [u_1, ..., u_m])$, since $\{u'_1, ..., u'_{i-1}, v'_{i-1}\}$ is now at most 1 vertex, $d_{out}(u'_i) = 1$ and in fact, $N_{out}(u'_i) = \{v_m\}$. Therefore, we may let $u_{m+1} = u'_i$ and $v_{m+1} = v_m$, contradicting maximality.

Case 2: $u'_i = v'_{i-1}$. Then, in $R(G, [u_1, ..., u_m])$, $\{u'_1, ..., u'_{i-1}, u'_i\}$ is now at most 1 vertex, and the vertex is u'_i. In fact, we can find k such that $v_k = u'_i$ and by linearity, we must have $u'_i = v_m$. Since $N_{out}(u'_i) \subseteq \{u'_1, ..., u'_{i-1}, v'_i\}$, we see that $d_{out}(u'_i) = 1$ and letting $u_{m+1} = u'_i = v_m$ contradicts maximality. □

Lemma 29. Let $u_1, ..., u_m$ be a maximal elimination ordering and $u'_1, ..., u'_l$ be another maximal elimination ordering with $u_1 = u'_1$. Then, $l = m$ and $u'_1, ..., u'_l$ is a permutation of $u_1, ..., u_m$. Furthermore, if $v_1, ..., v_m$ and $v_1, ..., v'_l$ are the corresponding out-sequences respectively, then $v_m = v'_l$.

Proof. Since $u_1 \in \{u'_1, ...u'_l\}$ and $u'_1 \in \{u_1, ..., u_m\}$, by Lemma 28, we see that $\{u_1, ..., u_m\}$ and $\{u'_1, ..., u'_l\}$ must contain each other and thus have the same elements. Hence $u'_1, ..., u'_l$ is a permutation of $u_1, ..., u_m$ and $l = m$.

Let $X = \{u_1, ...u_m\} = \{u'_1, ..., u'_l\}$. By Lemma 24, X hits v_m and X hits v'_l. Also, $v_m, v'_l \notin X$. Hence, $v_m = v'_l$. □

Lemma 30. If $u_1, ..., u_m$ and $w_1, ..., w_m$ are two elimination orderings that are permutations of each other, then $R(G, [u_1, ..., u_m]) = R(G, [w_1, ..., w_m])$.

5.2 Elimination Sets

Definition 31. Let $u_1, ..., u_m$ be the *unique maximal* (up to permutation) elimination ordering starting at $u_1 = u$ and let $v_1, ..., v_m$ be its out-sequence. We say that $S_u = \bigcup_{i=1}^{m} \{u_i, v_i\}$ is the **elimination set** of u. The unique "end vertex" v_m is called the **bad vertex** of S_u. We denote this bad vertex by $b_u := v_m$. Note that $S_u = \{v_m\} \cup \bigcup_{i=1}^{m} \{u_i\}$.

Definition 32. Let S_u be the elimination set of u, then $R(G, S_u)$ is defined to be the *unique* digraph (uniqueness is implied by Lemma 30) obtained by eliminating vertices according to a maximal elimination ordering starting at u.

In the rest of this paper, *all elimination orderings are maximal.*

Lemma 33. If $x \in S_u, y \in S_u^c$ and $(x, y) \in E(G)$, then $x = b_u$.

Proof. Lemma 24 implies that $S_u \setminus \{b_u\}$ hits $\{b_u\}$. The lemma follows. □

Lemma 34. Let S_u be the elimination set of u. Then, b_u has at least 2 out-neighbors in S_u^c. Furthermore, if $v \in S_u^c$ has out-neighbors in S_u, then v has out-neighbors in S_u^c.

Proof. By the maximality of the elimination ordering, b_u must have at least 2 out-neighbors in S_u^c. If $v \in S_u^c$ has out-neighbors in S_u, then v is an in-neighbor of b_u in $R(G, S_u)$. Again by the maximality of the elimination ordering v must have out-neighbors in S_u^c. □

Lemma 35. Let u be a vertex of out-degree 1 and v be its unique out-neighbor. Let w be another vertex. If S_w contains u or v, then $S_u \subseteq S_w$.

Proof. By maximality, we see that if S_w contains either u or v, then u must have been eliminated in the elimination ordering starting at w. Let $u_1, ..., u_m$ be the elimination ordering starting at $u_1 = u$. Therefore, by Lemma 28, we see that $\{u_1, ..., u_m\} \subseteq S_w$. By Lemma 24, $\{u_1, ..., u_m\}$ hits $\{v_m\}$, so we see that $v_m \in S_w$. We conclude that $S_u \subseteq S_w$. □

5.3 Elimination Decomposition

Let G be a strongly-connected digraph. Let $|V(G)| = n$ and $V(G) = \{w_1, \ldots, w_n\}$. Let $\{S_{w_1}, ..., S_{w_n}\}$ be the set of *all* elimination sets in G. Each elimination set S_{w_i} is a subset of $V(G)$. If there are sets $S_{w_i} = S_{w_j}$ then we remove one of them arbitrarily. Let $S = \{S_{v_1}, ..., S_{v_l}\}$ be the set of elimination sets without duplicates. The subset relation defines a natural partial order on the sets in S. Let $S_* = \{S_{u_1}, ..., S_{u_s}\}$ be the largest anti-chain of the sets in S.

Definition 36. We say that $S_* = \{S_{u_1}, ..., S_{u_s}\}$ is an **elimination decomposition** of G. We define $S_0 := V(G) \setminus \bigcup_{i=1}^{s} S_{u_i}$

As an example, we now exhibit an elimination decomposition of $G = S_{2,2,2}$, the spider graph with three legs of length two each. $S_{2,2,2}$ has three legs connected by a vertex (say w). Let u_i, v_i be the vertices in the i-th branch such that $(u_i, v_i), (v_i, u_i), (v_i, w), (w, v_i) \in E(G)$. Then, the elimination sets are $S_{u_i} = \{u_i, v_i, w\}$ for $1 \le i \le 3$. $S_* = \{S_{u_1}, S_{u_2}, S_{u_3}\}$ is the elimination decomposition of G.

Theorem 37 (Characterization Theorem)**.** *Let G be a strongly connected digraph. Then, G has directed pathwidth ≥ 2 if and only if $V(G)$ can be decomposed into sets $V_0, V_1, ..., V_s$, with $V(G) = \bigcup_{i=0}^{s} V_i$ and satisfying the following properties:*

- *Property 1: for $1 \le i \le s$, V_i is a proper subset of $V(G)$*
- *Property 2: V_0 is a set of vertices of out-degree ≥ 2*
- *Property 3: for $1 \le i \le s$, if $v_i \in V_i$ has out-neighbors in V_i^c, then v_i has at least 2 such out-neighbors*
- *Property 4: for $1 \le i \le s$, if $u \in V_i^c$ has out-neighbors in V_i, then u has out-neighbors in V_i^c.*

Proof. If G is of directed pathwidth ≥ 2, then consider its elimination decomposition $S_{u_1}, ..., S_{u_s}$ and let $V_i = S_{u_i}$ and $V_0 = S_0$. By definition of S_0, we see that $\bigcup_{i=1}^{s} V_i = V(G)$. Note that if there exists i such that $S_{v_i} = V(G)$, then G would have directed pathwidth ≤ 1, so S_{v_i} must be a strict subset. If there exists $v \in S_0$ that has outdegree ≤ 1, then S_v is an elimination set, but it is not necessarily maximal or in the elimination decomposition. However, we see that $S_v \subseteq S_w$ for some S_w in the elimination decomposition. But $v \in S_w$ contradicts $v \in S_0$. Properties 3 and 4 follow from Lemma 33 and Lemma 34.

If we can find such vertex sets $V_0, ...V_n$, then assume for contradiction that G is of pathwidth ≤ 1. Thus, since G is strongly connected, we can find a linear elimination ordering $u_1, ..., u_m$ and its outsequence $v_1, ..., v_m$ such that G is reduced to a single vertex. Since u_1 has outdegree 1, we see that $u_1 \in V_i$ for some $i \geq 1$. Since V_i is only a strict subset of $V(G)$, there must exists l minimal such that either $u_l \notin V_i$ or $v_l \notin V_i$.

Case 1: $u_l \notin V_i$. Since u_l can be contracted and the minimality of l, we know that $N_{out}(u_l) \subseteq \{u_1, ...u_{l-1}, v_{l-1}\} \subseteq V_i$. Note that u_l must have at least one out-neighbor in V_i, so this contradicts property 4.

Case 2: $v_l \notin V_i$. Since $u_l \in V_i$ and $v_l \notin V_i$, by property 3, u_l has at least 2 out-neighbors in V_i^c. Since $\{u_1, ..., u_l\} \in V_i$, we see that the two out-neighbors of u_l in V_i^c were not eliminated and u_l has outdegree ≥ 2 at time of elimination, a contradiction. □

6 Intersection Properties

6.1 Lonely Sets

Definition 38. Let $S_* = \{S_{u_1}, ..., S_{u_s}\}$ be an elimination decomposition of G. Then, $X \subseteq V(G)$ is **lonely** if there exists i such that $\bigcup_{j \neq i} S_{u_j} \cap X = \emptyset$

Lemma 39. *Let S_u be an elimination set in S_* and let v be the unique out-neighbor of u. Then, $\{u, v\}$ is lonely.*

6.2 Double Intersection

Lemma 40. *Let $S_{v_1}, ..., S_{v_t}$ be a subset of elimination sets in S_* such that S_{v_i} hits $S_{v_{i+1}}$ for $1 \leq i \leq t-1$, and S_{v_t} hits S_{v_1}. Then, $V(G) = \bigcup_{i=1}^{t} S_{v_i}$ and there are no other elimination sets in S_*.*

Proof. Let $H = \bigcup_{i=1}^{t} S_{v_i}$. For any vertex $v \in S_{v_i}$, all its out-neighbors are in $S_{v_i} \cup S_{v_{i+1}}$. Hence, H has no out-neighbors in H^c. By strong connectivity of G, $H^c = \emptyset$. □

Lemma 41. *If $S_u \cap S_v \neq \emptyset$, then $\{b_u, b_v\} \cap S_u \cap S_v \neq \emptyset$.*

Proof. By strong connectivity, $S_u \cap S_v$ must have out-neighbors in $(S_u \cap S_v)^c = S_u^c \cup S_v^c$. Without loss of generality, assume it has out-neighbors in S_u^c, but by Lemma 33, the only vertex in S_u that has out-neighbors in S_u^c is b_u. Therefore, $b_u \in S_u \cap S_v$. □

Corollary 42. *For an elimination set S_u, if $\{b_u\}$ is lonely, then S_u is lonely.*

Lemma 43. *If $b_u \in S_v$ and $b_u \neq b_v$, then S_u hits S_v.*

Proof. Consider the linear elimination order of S_v. Since $b_u \neq b_v$, we must have reduced b_u during the elimination process starting at v. Hence, $N_{out}(b_u) \subseteq S_v$. By Lemma 33, S_u hits S_v. □

Corollary 44. *If $S_u \cap S_v \neq \emptyset$ and $b_v \notin S_u$, then S_u hits S_v.*

Corollary 45. *If $b_u, b_v \in S_u \cap S_v$ and $b_u \neq b_v$, then S_u and S_v hit each other and $V(G) = S_u \cup S_v$.*

6.3 Triple Intersection

Now we consider the case when there are three distinct elimination sets S_u, S_v, S_w in S_* that pairwise intersect. We say that $S_u \cap S_v \cap S_w$ is the "triple intersection".

Lemma 46. *If S_u, S_v, S_w pairwise intersect and $S_u \cap S_v \cap S_w = \emptyset$, then S_u, S_v and S_w hit each other in a cycle and $V(G) = S_u \cup S_v \cup S_w$.*

Proof. Consider S_u, S_v and note that we cannot have $b_u, b_v \in S_u \cap S_v$ or else due to the empty triple intersection, we have $b_u, b_v \notin S_w$. But since S_w intersects both S_u, S_v, we must have $b_w \in S_u, S_v$. But, b_w is in the triple intersection, a contradiction. Since one of b_u, b_v must be in the intersection, so without loss of generality, let $b_u \in S_v$ and $b_v \notin S_u$. By Corollary 44, S_u hits S_v.

Simiarly, S_w intersects S_u, but since $b_u \in S_u \cap S_v$ and the triple intersection is empty, we conclude that $b_u \notin S_w$. By Corollary 44, we conclude that S_w hits S_u and by Lemma 41, $b_w \in S_u$. Lastly note that $b_w \notin S_v$ since the triple intersection is empty. We conclude that S_v hits S_w. Thus, we have 3 elimination sets that hit each other and by Lemma 40, $V(G) = S_u \cup S_v \cup S_w$. □

Lemma 47. *If $S_u \cap S_v \cap S_w \neq \emptyset$, then one of b_u, b_v, b_w must be in the triple intersection.*

Proof. Let $H = S_u \cap S_v \cap S_w$. By strong connectivity of G, H must have an out-neighbor in H^c. Let $u \in H, v \in H^c$ such that $(u, v) \in E(G)$. Note that either $v \in S_u^c$ or $v \in S_v^c$ or $v \in S_w^c$. By Lemma 33, we conclude that u must be either b_u or b_v or b_w respectively. □

Lemma 48. *If we have S_u, S_v, S_w such that $b_u, b_v, b_w \in S_u \cap S_v \cap S_w$, then $b_u = b_v = b_w$.*

Proof. By Corollary 45, if any of the bad vertices are not equal, then we have only two elimination sets, contradicting the existence of S_u, S_v and S_w. □

Lemma 49. *If we have S_u, S_v, S_w such that $S_u \cap S_v \cap S_w \neq \emptyset$ and b_u is in the triple intersection and b_v is not in the triple intersection, then S_u hits S_v and*

1) If $b_u = b_w$, then S_w hits S_v

2) If $b_w = b_v$, then S_u hits S_w

3) If all vertices are distinct, then S_u hits S_w. Also, S_w hits S_v or S_v hits S_w.

Proof. Assume b_u is in the triple intersection set $H = S_u \cap S_v \cap S_w$ and b_v is not. Note first that $b_v \notin S_u$, or else by Lemma 43, $b_u \neq b_v$ implies we only have two elimination sets, contradicting existence of S_w. By Corollary 44, S_u hits S_v.

If $b_u = b_w$, then $b_w \in H$ and $b_v \neq b_w$. As previous, we conclude that S_w hits S_v.

If $b_w = b_v$, then we note that $b_w \neq b_u$, so S_u hits S_w.

Otherwise, we see that b_u, b_v, b_w are all distinct vertices and note also by Lemma 43, $b_v, b_w \notin S_u$. $b_w \neq b_u$ still gives S_u hits S_w. Then, one of b_v, b_w must be in $S_v \cap S_w$, so either S_v hits S_w or vice versa. □

Lemma 50. *If we have S_u, S_v, S_w such that $S_u \cap S_v \cap S_w \neq \emptyset$ and S_u hits S_v, then one of the following holds:*

1) b_u is in the triple intersection and b_v is not

2) b_w is in the triple intersection and b_u, b_w are not and S_w hits S_u, S_v.

Proof. Note that if $b_u = b_v$, then since S_u hits S_v, all the out-neighbors of $b_u = b_v$ are contained in $S_u \cup S_v$, which means S_u, S_v hit each other and by Lemma 40, this contradicts existence of S_w. Also, if $b_v \in S_u$, then $b_u \neq b_v$ implies S_v hits S_u, which again contradicts existence of S_w.

Therefore, $b_v \notin S_u$ and so b_v is not in the triple intersection. If b_u is in the triple intersection, we are done. Otherwise, b_u, b_v are both not in the triple intersection and by Lemma 47, b_w is in the triple intersection. By Lemma 49, S_w hits both S_u, S_v. □

6.4 More Triple Intersections

In this section, we study intersections of the form $S_u \cap S_v \cap S_w^c$, under the assumption that the triple intersection is not empty.

Lemma 51. *Let $S_u \cap S_v \neq \emptyset$ and let $u_1, ..., u_m$ be the elimination ordering of S_u and $v_1, .., v_m$ be the corresponding out-sequence. If $u_l = b_v$ and there exists $t \geq l$ such that $v_t \in S_v$, then S_u, S_v hit each other and $V(G) = S_u \cup S_v$.*

Proof. We see that by Lemma 24, $\{u_1, ..., u_t\}$ hits v_t in G. Since $u_l = b_v$ and $v_t \in S_v$, we see that by Lemma 33, $\{u_1, ..., u_t\} \cup S_v$ has no out-neighbors to its complement. So, by strong connectivity, $G = \{u_1, ..., u_t\} \cup S_v$. Therefore, $V(G) = S_u \cup S_v$ and there are no other vertices. Since there are only two elimination sets, S_u, S_v hit each other trivially. □

Lemma 52. *Let $S_u \cap S_v \neq \emptyset$ and let $u_1, ..., u_m$ be the elimination ordering of S_u and $v_1, .., v_m$ be the corresponding out-sequence. Assume there exists t minimal such that $u_t \in S_v$ and there exists minimal $p \geq t$ such that $v_p \notin S_v$. If $(S_u \cup S_v)^c \neq \emptyset$, then $u_p = b_v$ and $b_u \notin S_v$.*

Proof. We see that by definition, $v_l \in S_v$ for $t \leq l < p$. Now, by linearity, we see that either $v_{p-1} = v_p$ or $v_{p-1} = u_p$. Since $v_{p-1} \in S_v$, $v_{p-1} = u_p$ must hold. Realize b_v cannot be eliminated before u_p, since if $p = t$, then minimality of t is contradicted and if $p > t$, since $v_{p-1} \in S_v$, by Lemma 51, we see that $(S_u \cup S_v)^c = \emptyset$, a contradiction.

Since $u_p \in S_{v_j}$ and $v_p \notin S_{v_j}$ and b_v was not eliminated, we conclude that $u_p = b_v$. Also, we claim $b_u \notin S_v$. If $b_u \in S_v$ and since $b_v \in S_u$, then if $b_u \neq b_v$, we have S_u, S_v hit each other. By Lemma 40, this contradicts $(S_u \cup S_v)^c = \emptyset$. Otherwise, $b_u = b_v$, but that is impossible since b_u cannot be reduced in the elimination ordering of S_u. □

Lemma 53. *If $b_u \in S_u \cap S_v \cap S_w$ and let $u_1, ..., u_m$ be the elimination ordering of S_v and $v_1, .., v_m$ be the corresponding out-elimination ordering. Assume there exists t minimal such that $u_t \in S_u \cap S_w^c$. Then, $v_t \notin S_w$ and u_t has out-neighbors in $S_u \cap S_v \cap S_w^c$.*

Proof. Consider $N_{out}(u_t)$ and note that $N_{out}(u_t) \subseteq \{u_1, ..., u_m, v_t\}$, so we know that $N_{out}(u_t) \subseteq S_v$. By Lemma 33, since $u_t \neq b_u$, we also know $N_{out}(u_t) \subseteq S_u$. Next, note that u_t cannot only have out-neighbors in S_w, or else $u_t \in S_w$. So, u_t must have out-neighbors in $S_u \cap S_v \cap S_w^c$.

Now, if $v_t \in S_w$, then note that b_w could not have been previously reduced by Lemma 51. Therefore, if v_t has any out-neighbors in S_w, they cannot have been reduced to its out-neighbors in S_w^c. Since u_t has out-neighbors in $S_u \cap S_v \cap S_w^c$, they must have been reduced previously in $u_1, ..., u_{t-1}$, contradicting minimality of t. □

Lemma 54. *Let $G \in \mathcal{F}$. If $b_u \in S_u \cap S_v \cap S_w$, then either $S_u \cap S_v^c \cap S_w = \emptyset$ or $S_u \cap S_v \cap S_w^c = \emptyset$.*

Proof. Assume otherwise that both sets are non-empty. Consider the elimination order of S_u, $u_1, ..., u_m$ and its corresponding out-sequence $v_1, .., v_m$, where $u_1 = u$ and $v_m = b_u$. Note that $u_1 \in (S_v \cup S_w)^c$ and we know that since $b_u \notin S_u \cap S_v^c \cap S_w, S_u \cap S_v \cap S_w^c$, there must exist minimal $l > 1$ such that $u_l \in S_v \cup S_w$.

Without loss of generality, we may assume that $u_l \in S_v$. Assume that there exists a minimal $r \geq l$ such that $v_r \notin S_v$. But $(S_u \cup S_v)^c \neq \emptyset$ and $b_u \in S_v$, which contradicts Lemma 52.

So, no such r exists. Now, since $b_u \notin S_u \cap S_w \cap S_v^c$ and it is a non-empty set, we can find a minimal $q > l$ such that $u_q \in S_u \cap S_w \cap S_v^c$. Note $v_q \in S_v$, which contradicts Lemma 53. □

Lemma 55. *Let G be a minimal forbidden minor. If $b_u \in S_u \cap S_v \cap S_w$ and b_v, b_w are not, then either $S_u \cap S_v^c \cap S_w = \emptyset$ or $S_u^c \cap S_v \cap S_w = \emptyset$.*

Proof. Since $b_u \neq b_v, b_w$, first note that the out-neighbors of b_u in S_u^c are contained in $S_v \cap S_w$ and there are at least 2 such out-neighbors. Assume that both sets are non-empty. Consider the elimination order of S_w, $u_1, ...u_m$, and its corresponding out-sequence $v_1, ..., v_m$, where $u_1 = w, v_m = b_w$. Note that $u_1 \in (S_u \cup S_v)^c, b_u \in S_u \cup S_v$, so we can find minimal $l > 1$ such that $u_l \in S_u \cup S_v$.

Case 1: $u_l \in S_u$. Note that $b_w \notin S_u$, or else S_u, S_w will hit one another. So, there must exist minimal $t \geq l$ such that $v_t \notin S_u$. By Lemma 52, $u_t = b_u$. But b_u has at least two out-neighbors in $S_u^c \cap S_v \cap S_w$ and so one of its out-neighbors must have been previously reduced and there exists a minimal $q < t$ such that $u_q \in S_u^c \cap S_v \cap S_w$. We see that $q > l$.

So, $u_q \notin S_u, v_q \in S_u$, thus in G u_q has an out-neighbor $x \in S_u^c$ in G that must have been previously reduced. We see that if $x \in S_{v_j}$, then the minimality of y is contradicted. We must have $x \notin S_v$ and since $u_q \in S_v$, $u_q = b_v$ by Lemma 33 and $b_v \in S_w$.

Note that $v_{t-1} = u_t = b_u$ is forced since $v_{t-1} = v_t$ contradicts the minimality of t. But, $v_{t-1} = b_u \in S_v$ and $t - 1 \geq q$, where $u_q = b_v$, contradicting Lemma 51.

Case 2: $u_l \in S_v \cap S_u^c$. Since $S_u \cap S_v^c \cap S_w \neq \emptyset$, we see that there exists a minimal $t \geq l$ such that $u_t \in S_v^c \cap S_u$ or $v_t \in S_v^c \cap S_u$.

Case 2.1: $u_t \in S_v^c \cap S_u$. Then by Lemma 53, u_t has an out-neighbor x in G in S_v^c. Note either x was previously reduced or exists $k < t$ such that $v_k = x$. Either case, $x \in S_u$ contradicts the minimality of t. So $x \notin S_u$, but $u_t \neq b_u$ contradicts Lemma 33.

Case 2.2: $v_t \in S_v^c \cap S_u$. Let $q \leq t$ be minimal such that $v_q \in S_v^c$. By Lemma 52, we see that $u_q = b_v$. If b_u was reduced on u_t or before, then $v_t \in S_{v_i}$ contradicts Lemma 51.

Since $b_u \neq b_w$, there must exist a minimal $r > t$ such that $u_r = b_u$. Note that if $v_{r-1} = b_u \in S_v$, then since b_v is already contracted, this contradicts Lemma 51. Therefore, $v_{r-1} = v_r$. However, we claim that $v_{r-1} \in S_u$. Assume not and consider the minimal $t < p < r$ such that $v_p \notin S_u$. Then, we see that $v_{p-1} = u_p$ and so, $u_p \in S_u$. However, since b_u is not yet reduced, this forces $u_p = b_u$, a contradiction to the minimality of r. So, $v_r = v_{r-1} \in S_u$. But $u_r = b_u$ and $v_r \in S_u$ contradicts Lemma 51. □

Lemma 56. *If $b_u, b_v \in S_u \cap S_v \cap S_w$, then $S_u \cap S_v^c \cap S_w = \emptyset$ or $S_u^c \cap S_v \cap S_w = \emptyset$.*

Proof. Note that if b_w is in the triple intersection, then this follows from Lemma 54. So, we may assume b_w is in the triple intersection.

Assume that both sets are nonempty. Consider the elimination ordering of S_w as $u_1, ..., u_m$ and its corresponding out-sequence $v_1, .., v_m$. Now, since $b_u \in S_u \cap S_v \cap S_w$, we can find a minimal l

such that $u_l \in S_u \cup S_v$. Without loss of generality, let $u_l \in S_u$. Now, note that $b_w \in (S_u \cup S_v)^c$, or else we get two sets that hit each other by Corollary 45. So, since $S_u^c \cap S_v \cap S_w \neq \emptyset$, we can find minimal $q \geq l$ such that $u_q \in S_u^c \cap S_v$ or $v_q \in S_u^c \cap S_v$.

Case 1: $u_q \in S_u^c \cap S_v$. Then by Lemma 53, u_q has an out-neighbor x in S_u^c. Note that the out-neighbor cannot be in S_v since x must be previously reduced or reduced to, contradicting minimality. Thus, since $u_q \neq b_u$, we have a contradiction.

Case 2: $v_q \in S_u^c \cap S_v$. By Lemma 52, $u_q = b_v = b_j$ and $v_q \in S_v$, contradicting Lemma 51. ☐

Lemma 57 (Non-Intersection Theorem). *If $S_u \cap S_v \cap S_w \neq \emptyset$, then $S_u \cap S_v^c \cap S_w = \emptyset$ or $S_u \cap S_v \cap S_w^c = \emptyset$.*

Proof. If b_u is in the triple intersection, then it follows by Lemma 54. If b_u is not in the triple intersection, but both b_v, b_w are, then it follows by Lemma 56. If both b_u, b_v are not in the triple intersection, or if b_u, b_w are not in the triple intersection, then it follows by Lemma 55. ☐

6.5 Applications of Intersection Theorems

Definition 58. S_u is an **A-type** (elimination set) if there exists an elimination set S_v such that S_u hits S_v. If S_u is not an A-type, we call it a **non A-type**.

Lemma 59. *If S_u, S_v are non A-types and $S_u \cap S_v \neq \emptyset$, then $b_u, b_v \in S_u \cap S_v$ and $b_u = b_v$.*

Proof. Note that if $b_u \notin S_u \cap S_v$, then by Corollary 44, S_v must hit S_u, contradicting that S_u are non A-types. Therefore, $b_u, b_v \in S_u \cap S_v$ and by Lemma 43, $b_u = b_v$. ☐

Lemma 60. *If S_u hits S_v and S_w hits S_x and $b_u = b_w$, then either S_u hits S_x or S_w hits S_v. Also, $S_v \cap S_x \neq \emptyset$.*

Proof. Assume that S_w does not hit S_v, then there is an out-neighbor of $b_w = b_u$, v_1, such that $v_1 \notin S_w \cup S_v$. But S_u hits S_v, so $v_1 \in S_u \cup S_v$ and therefore we conclude that $v_1 \in S_u \cap S_v^c$. Symmetrically, we see that $v_1 \in S_x \cap S_w^c$. Therefore, $v_1 \in S_u \cap S_x$. Now, we split into cases.

Case 1: $b_u \notin S_x$. By Corollary 44, S_x hits S_u and $b_x \in S_u$. If $b_x \notin S_v$, then again, S_v hits S_x. But now, S_u, S_v, S_x hit each other in a cycle, contradicting that existence of S_w. Lemma 40 So, $b_x \in S_v$. Now, we see $b_x \in S_u \cap S_v$ and so by Corollary 44, we must have $b_u \in S_v$ or else S_v would hit S_u, a contradiction.

In this case, we have that $b_u \in S_u \cap S_v \cap S_w$ and S_u hits S_v. Note that if $b_u = b_v$, then S_u, S_v hits each other, contradicting the existence of S_w. Thus $b_u \neq b_v$, and so $N_{out}(b_u) \subseteq S_v$. Therefore, S_w hits S_v, a contradiction.

Case 2: $b_u \in S_x$. Then, $b_u = b_w \in S_u \cap S_w \cap S_x$ and since S_w hits S_x, we can repeat the same argument above to show that $N_{out}(b_w) \subseteq S_x$. In this case, S_u hits S_x.

So, in either case there exists an elimination set S that hits both S_v, S_x. Therefore, we conclude that $S_v \cap S_x \neq \emptyset$. ☐

Lemma 61. *If S_u hits S_v and S_w is a non A-type and $b_u = b_w$, then $S_u \cap S_v = \emptyset$.*

Proof. Assume that $S_u \cap S_v \neq \emptyset$, then S_v cannot hit S_u, or else S_u, S_v hit each other. So, $b_u \in S_v$. Therefore, $b_u \in S_u \cap S_v \cap S_w$ and S_u hits S_v and S_w is a non A-type, so by Lemma 50, we conclude that b_u is in the triple intersection and b_v is not. However, since $b_u = b_w$, by Lemma 49, S_w must be an A-type that hits S_v, contradicting the fact that S_w is a non A-type. ☐

7 Minimal Forbidden Minor Properties

Let $G \in \mathcal{F}$. As noted before, G is strongly-connected and $\mathsf{dpw}(G) \geq 2$. We say that S_u **shares its bad vertex** with S_v if $b_u = b_v$.

7.1 A-Types In A Cycle

When there are A-types that hit each other in a cycle, we firstly want to be able to bound the length of the cycle or the number of A-types in the cycle. Using minimality, we show that the number of A-types can be at most 4.

Theorem 62. *Let $G \in \mathcal{F}$. If there exists a number of A-types that hit each other in a cycle, then there are at most 4 such A-types.*

Proof. Let $S_{v_1}, ..., S_{v_n}$ be A-types such that S_{v_i} hits $S_{v_{i+1}}$, where $v_{n+1} = v_1$, for $1 \leq i \leq n$ and $n > 4$. By Lemma 40, we know that no other vertices or maximal elimination sets exist. Furthermore, we may assume that n is minimal. Therefore, for any i, S_{v_i} can only hit $S_{v_{i+1}}$, or else we can find a smaller cycle of A-types that hit each other, a contradiction to minimality. Similarly, $S_{v_{i+1}}$ can only be hit by S_{v_i}.

Assume that we have more than 4 A-types. Then, our first claim is for any i, S_{v_m} cannot intersect both $S_{v_i}, S_{v_{i+1}}$ for $m \neq i, i+1$. Assume, for contradiction, that $S_{v_m} \cap S_{v_i} \neq \emptyset, S_{v_m} \cap S_{v_{i+1}} \neq \emptyset$.

Case 1: $S_{v_i} \cap S_{v_{i+1}} = \emptyset$. By Lemma 44, since S_{v_m} cannot hit $S_{v_{i+1}}$, $b_{v_{i+1}} \in S_{v_m}$. Similarly, S_{v_i} cannot hit S_{v_m}, so $b_{v_m} \in S_{v_i}$. Since $S_{v_i} \cap S_{v_{i+1}} = \emptyset$, we deduce $b_{v_m} \notin S_{v_{i+1}}$. So, $S_{v_{i+1}}$ must hit S_{v_m}. If $b_{v_i} \notin S_{v_m}$, then S_{v_m} must hit S_{v_i}, so there are at most 3 A-types, a contradiction. So, $b_{v_i} \in S_{v_m}$. By Lemma 45, $b_{v_i} = b_{v_m}$ and $N_{out}(b_{v_m}) = N_{out}(b_{v_i}) \subseteq S_{v_i} \cup S_{v_{i+1}}$. Then, $H = S_{v_i} \cup S_{v_{i+1}} \cup S_{v_m}$ has no out-neighbors in H^c, so $G = H$ only has 3 elimination sets, a contradiction.

Case 2: $S_{v_i} \cap S_{v_{i+1}} \neq \emptyset$. Then all $S_{v_i}, S_{v_{i+1}}, S_{v_m}$ pairwise intersect and note that if $S_{v_m} \cap S_{v_i} \cap S_{v_{i+1}} = \emptyset$, then by Lemma 46, we have 3 A-types that hit each other, a contradiction. So, $S_{v_m} \cap S_{v_i} \cap S_{v_{i+1}} \neq \emptyset$, and S_{v_i} hits $S_{v_{i+1}}$, so by Lemma 50, either b_{v_i} is in the triple intersection and $b_{v_{i+1}}$ is not, or S_{v_m} hits both $S_{v_i}, S_{v_{i+1}}$. As noted before, the second case cannot happen, so we must have b_{v_i} is in the triple intersection and $b_{v_{i+1}}$ is not. By Lemma 49, we see that we must have either S_{v_m} hits $S_{v_{i+1}}$ or S_{v_i} hits S_{v_m}, both contradictions. So, our claim holds.

Our second claim is if S_{v_i} intersects S_{v_j}, then $j = i, i-1, i+1$, where $v_0 = v_n, v_1 = v_{n+1}$. Assume that S_{v_i} and S_{v_j} intersects and $j \neq i, i-1, i+1$. Then, we cannot have S_{v_i} hits S_{v_j} or S_{v_j} hits S_{v_i} since $j \neq i, i-1, i+1$. Therefore, by Lemma 44, $b_{v_i}, b_{v_j} \in S_{v_i} \cap S_{v_j}$ and since we have more than 2 elimination sets, $b_{v_i} = b_{v_j}$ must hold. So, by Lemma 60, we know that S_{v_i} hits $S_{v_{j+1}}$, or S_{v_j} hits $S_{v_{i+1}}$, both of which are impossible since $j \neq i, i-1, i+1$.

Finally, we proceed with our proof. First, assume that we can find an A-type S_{v_i}, such that v_i is not an in-neighbor of b_{v_i}. Without loss of generality, let S_{v_2} be our A-type that satisfies this. Then, note that we may reduce v_2 in S_{v_2} to get S'_{v_2} and consider the graph $G' = R(G, v_2)$. It is strongly connected and so we claim that we can appeal to the characterization theorem with $V_0 = S_0 = \emptyset, V_1 = S_{v_1} \cup S'_{v_2}, V_2 = S'_{v_2} \cup S_{v_3}$ and $V_i = S_{v_{i+1}}$ for $3 \leq i \leq n-1$. Note that all V_i are strict subsets of G for $3 \leq i \leq n-1$, trivially. Furthermore, V_1, V_2 are also strict subsets since they do not contain the lonely vertices in S_{v_4}.

Clearly, we still have $\bigcup_{i=0}^{n-1} V_i = V(G')$. Now, note that v_2 and its out-neighbor w_2 are lonely vertices in S_{v_2} and are not in any other elimination sets. So, for $i \geq 3$, note that $V_i = S_{v_{i+1}}$ does not have v_2, w_2 as out-neighbors. This is because if $S_{v_{i+1}}$ does, then since $S_{v_{i+1}}$ hits $S_{v_{i+2}}$, we see that either $v_2, w_2 \in S_{v_{i+1}} \cup S_{v_{i+2}}$. But, by the loneliness of v_2, w_2, one of $S_{v_{i+1}}, S_{v_{i+2}}$ must be S_{v_2}, which is impossible.

So, in G', $b_{v_{i+1}} \in V_i$ is still the unique vertex with at least 2 edges from v_i to V_i^c. For V_1, we claim that b_2 also satisfies this constraint. Note that in G', there are two out-neighbors of b_2 that are in $S_{v_2}^c$, however, the problem is that they could be in S_{v_1}. Assume w is an out-neighbor of b_2 in S_{v_1}, then note that w must be in S_{v_3} since S_{v_2} hits S_{v_3}. But, S_{v_1} cannot intersect S_{v_3} when there are more than 3 A-types, by our second claim. Similarly, this holds for V_2.

Lastly, we claim that property 4 holds. If $u \in V_1^c$ and there exists $v \in V_1$ such that $(u, v) \in E(G')$, then in G, we know that $v \in S_{v_1}$ or $v \in S_{v_2}$, and $u \notin S_{v_1}, S_{v_2}$.

Case 1: $v \in S_{v_1}$. Then, we can find $w \in S_{v_1}^c$ such that $(u, w) \in E(G)$. If $w \notin S_{v_2}$, then note that $w \notin S_{v_1} \cup S_{v_2}$ and $(u, w) \in E(G')$, so we are done.

If $w \in S_{v_2}$, then we see that since $S_0 = \emptyset$, we claim that u must be a bad vertex of another A-type, S_{v_k} and clearly $S_{v_k} \neq S_{v_1}, S_{v_2}$. If u is not the bad vertex, then there must be an elimination set S_{v_m} that contains u and its out-neighbors. But, S_{v_m} intersects S_{v_1}, S_{v_2}, a contradiction.

So, there must exist an A-type S_{v_k} such that $u = b_{v_k}$. Note that S_{v_k} hits $S_{v_{k+1}}$ and $v, w \in S_{v_k} \cup S_{v_{k+1}}$. If both $v, w \in S_{v_k}$, then note that S_{v_k} intersects both S_{v_1}, S_{v_2}, a contradiction to our first claim.

If $v \in S_{v_k}$, then S_{v_k} must be $S_{v_0} = S_{v_n}$, by our second claim. Then, we know that $S_{v_{k+1}} = S_{v_1}$. But, $w \in S_{v_{k+1}} = S_{v_1}$, which contradicts $w \notin S_{v_1}$.

So we must have $w \in S_{v_k}$ and by claim 2, we must have $S_{v_k} = S_{v_3}$. Since $v \in S_{v_{k+1}}$, we must have that $S_{v_{k+1}} = S_{v_0} = S_{v_n}$. So, we see that $n = 4$, contradiction.

Case 2: $v \in S_{v_2}$. This case is symmetric to case 1.

Therefore, V_1 satisfies property 4. Similarly, we may repeat the same argument for V_2.

For $i \geq 3$, if $u \in V_i^c$ and there exists $v \in V_i$ such that $(u, v) \in E(G')$, then note that $(u, v) \in E(G)$ so we can find $w \in V_i^c$ such that $(u, w) \in E(G)$. We see that $w \in V_i^c$ in G' and $(u, w) \in E(G')$, unless $w = v_2$ and $u = w_2$. Note that by choice, $u = w_2$ is not the bad vertex, so we must have $v \in S_{v_2} \cap V_i$ in G, contradicting claim 2.

So, all properties are satisfied and we may use the characterization theorem and get a contradiction to minimality. Therefore, G has at most 4 A-types that hit each other in a a cycle.

Note that if we cannot find an A-type S_{v_i} such that v_i is not an in-neighbor of b_{v_i}, then all b_{v_i} is lonely. By Lemma 42, all S_{v_i} are lonely. Now, we may repeat the argument above, except during the last paragraph:

For $i \geq 3$, if $u \in V_i^c$ and there exists $v \in V_i$ such that $(u, v) \in E(G')$, then note that $(u, v) \in E(G)$ so we can find $w \in V_i^c$ such that $(u, w) \in E(G)$. We see that $w \in V_i^c$ in G' and $(u, w) \in E(G')$, unless $w = v_2$ and $u = w_2 = b_{v_2}$. However, $N_{out}(b_{v_2}) \subseteq S_{v_2} \cup S_{v_3}$, we have that V_i either intersects S_{v_2} or S_{v_3}, contradicing that V_i is lonely. □

7.2 Connected Sets

Recall that S_u shares its bad vertex with S_v if $b_u = b_v$. Note that S_u shares its bad vertex with itself.

Definition 63. An elimination set S_u is **connected** to a *non A-type* S_v with a **connection of length** L if there exists elimination sets $S_{v_1}, ..., S_{v_{L+1}}$ such that $S_u = S_{v_1}$ and S_v shares its bad vertex with $S_{v_{L+1}}$ and for $1 \leq i \leq L$, S_{v_i} shares its bad vertex with an elimination set S_{w_i} such that S_{w_i} hits $S_{v_{i+1}}$.

It is important to note that an elimination set can only be connected to a non A-type elimination set. Connection to an A-type elimination set is not defined and is omitted for simplicity. Note that an elimination set S_u could have multiple connections to multiple non A-types.

If S_u is a non A-type, then S_u is connected to itself with length 0. Also, any A-type S_u is connected to some non A-type S_v because there are no cycle of A-types that hit each other. Indeed, if S_u hits a non A-type S_v, then S_u is connected to S_v. If not, S_u hits another A-type, which must hit another elimination set. Eventually an A-type must hit a non A-type, S_v and so S_u is connected to S_v. A non A-type elimination set is connected to itself and possibly others if it shares its bad vertex with another elimination set.

Observation 64. *For any elimination set S_u, there exists S_v such that S_u is connected to S_v.*

Definition 65. Let S_u be a non A-type in G and let $S_{v_1}, ..., S_{v_k}$ be all elimination sets connected to S_u. The **connected set** of S_u is $C_u = \bigcup_{i=1}^{k} S_{v_i}$

If S_u is connected to S_v and S_v is connected to S_w, then S_u is connected to S_w by simply concatenating the connections. Similarly, if S_u hits S_v and S_v is connected to S_w with length L, then S_u is connected to S_w with length $L+1$. So these "connections" define a quasi-order, i.e., they are reflexive and transitive.

Lemma 66. *If $(u,v) \in E(G)$ and $u \in C_w, v \in C_w^c$, then $u = b_w$.*

Proof. Assume that $(u,v) \in E(G)$ and $u \in C_w, v \in C_w^c$. There exists some S_x such that $u \in S_x \subseteq C_w$. By Lemma 33, we see that $u = b_x$ is forced. If S_x is connected to S_w with length 0, then $b_x = b_w$. Therefore, $u = b_x = b_w$.

Otherwise, S_x is connected to S_w with length ≥ 1, so there exists $S_{v_1}, ..., S_{v_{L+1}}$ such that $S_x = S_{v_1}$ and S_v shares its bad vertex with $S_{v_{L+1}}$ and for $1 \leq i \leq L$, S_{v_i} shares its bad vertex with an elimination set S_{w_i} such that S_{w_i} hits $S_{v_{i+1}}$. By definition, it is clear that S_{w_1}, S_{v_2} also is connected to S_w and so $S_{w_1}, S_{v_2} \subseteq C_w$. Since $b_{w_1} = b_x$ and S_{w_1} hits S_{v_2}, we conclude that $N_{out}(b_x) \subseteq S_{w_1} \bigcup S_{v_2} \subseteq C_w$, contradicting $v \in C_w^c$. ☐

Lemma 67. *If an A-type S_u is connected to S_v with length at least $L \geq 1$ and S_u hits S_w, then $S_w \subseteq C_v$ and S_w is connected to S_v with length $\leq L$.*

Proof. Since S_u is connected to S_v with length $L \geq 1$, we see that there are maximal elimination sets $S_{v_1}, ...S_{v_{L+1}}$ such that $S_u = S_{v_1}$ and S_v shares its bad vertex with $S_{v_{L+1}}$ and for $1 \leq i \leq L$, S_{v_i} shares its bad vertex with an elimination set S_{w_i} such that S_{w_i} hits $S_{v_{i+1}}$.

First note that S_u shares its bad vertex with S_{w_1} that hits S_{v_2}, where S_{v_2} shares its bad vertex with S_{w_2}. Since S_u hits S_w and $b_u = b_{w_1}$, by Lemma 60, $S_w \cap S_{v_2} \neq \emptyset$. Then, if S_w hits S_{v_2}, then S_w is connected to S_v with length L. $S_w \subseteq C_v$. So, we conclude that $b_{v_2} \in S_w$.

Now, we will proceed by induction on L. For the base case $L = 1$, note $b_{v_2} = b_v$, so $S_v \cap S_w \neq \emptyset$. Since S_v is a non A-type, by Corollary 44, we must have $b_w \in S_v$. Now, by Corollary 45, we conclude that $b_w = b_v$ and $S_w \subseteq C_v$ with length $L - 1$.

Now, we induct on L and assume that our result holds for $L = i \geq 1$. Then, let $L = i + 1$. Note that if S_{v_2} hits S_w, then we are done by the inductive hypothesis since there is a connection from S_{v_2} to S_v with length i. Otherwise, S_{v_2} does not hit S_w, so $b_w \in S_{v_2}$. By Corollary 45, $b_w = b_{v_2} = b_{w_2}$ and so, $S_w \subseteq C_v$ and S_v is connected to S_w with length $\leq L$ ☐

Lemma 68. *Let S_u be any elimination set and $S_u \cap C_v \neq \emptyset$, then $S_u \subseteq C_v$.*

Proof. If $S_u \cap C_v \neq \emptyset$, then there exists some elimination set $S_w \subseteq C_v$ such that $S_w \cap S_u \neq \emptyset$. If S_u hits S_w, then $S_u \subseteq C_v$. Otherwise, $b_w \in S_u$. Also, if $b_u = b_w$, then $S_u \subseteq C_v$. Therefore, we conclude that $b_u \notin S_w$ and S_w hits S_u. Now, we do casework based on whether S_w shares its bad vertex as S_v.

Case 1: $b_w = b_v$. Since S_v is a non A-type and S_w hits S_u, by Lemma 61, $S_w \cap S_u = \emptyset$, a contradiction.

Case 2: $b_w \neq b_v$. Then, S_w is connected to S_v with length at least 1 and S_w hits S_u, so $S_u \subseteq C_v$ by Lemma 67. □

Lemma 69. *Let C_u, C_v be connected sets and $C_u \cap C_v \neq \emptyset$, then either $C_u \subseteq C_v$ or $C_v \subseteq C_u$.*

Proof. Let $w \in C_u \cap C_v$, then by Lemma 68, note that any elimination set that contains w will be in C_u, C_v. So, we can find some elimination set $S_x \subseteq C_u, C_v$. As before, note that either $b_x = b_u$ or $b_x \neq b_u$.

Case 1: $b_x = b_u$. Then $S_u \cap C_v \neq \emptyset$ and so, $S_u \subseteq C_v$. Since S_u is connected to S_v, we conclude that any set in S_u is also connected to S_v, by transitivity. So, $C_u \subseteq C_v$.

Case 2: $b_x \neq b_u$. Note that S_x must share its bad vertex with some S_y that is an A-type connected to S_u. But then, $S_u \cap C_v \neq \emptyset$ and $S_u \subseteq C_v$. If S_u shares a bad vertex with S_v, then repeating the argument in case 1 gives $C_v \subseteq C_u$.

Otherwise, S_u is an A-type of length at least 1 in both C_u, C_v, so it hits S'_u and by Lemma 67, $S'_u \subseteq C_u, C_v$. Then, repeat the argument until we see either S_u, S_v and deduce that either $C_u \subseteq C_v$ or vice versa. □

7.3 Minimally Connected Sets

Definition 70. Let C_u be the connected set of S_u and let $C_{v_1}, ..., C_{v_k}$ be all the *properly* contained connected sets of C_u. The **minimally connected set** of S_u is $M_u = C_u \cap \bigcap_{i=1}^k C_{v_i}^c$.

Note that by definition, if $M_{u_1}, ..., M_{u_k}$ are all the minimally connected sets and $C_{u_1}, ..., C_{u_k}$ are all the connected sets, then

$$\bigcup_{i=1}^k M_{u_i} = \bigcup_{i=1}^k C_{u_i} = S_0^c$$

The last equality holds because every elimination set is connected to some elimination set.

Lemma 71. *If $M_u \cap M_v \neq \emptyset$, then $M_u = M_v$.*

Proof. Note that $C_u \cap C_v \neq \emptyset$. By Lemma 69, either $C_u \subseteq C_v$ or $C_v \subseteq C_u$. Without loss of generality, $C_u \subseteq C_v$. If $C_u \subsetneq C_v$, then $M_v \cap C_u = \emptyset$, a contradiction. Therefore, $C_u = C_v$ and by definition, $M_u = M_v$. □

Minimally connected sets that are not equal do not intersect at all, allowing us to isolate the effects of minor operations in one minimally connected set.

Definition 72. If S_u is connected to a non A-type S_v, then the **distance** from S_u to S_v is the *minimum* $L \geq 0$ such that S_u has a connection of length L to S_v. We denote this by $d(S_u, S_v)$.

Definition 73. S_u is **minimally connected** to S_v if there does not exist S_w such that $d(S_u, S_w) < d(S_u, S_v)$.

Lemma 74. If S_u is minimally connected to S_v and $d(S_u, S_v) > 0$, then there are no non A-types that share the same vertex as S_u.

Proof. If S_w is a non A-type and $b_w = b_u$, then $d(S_u, S_w) = 0$ is a contradiction of minimality. ☐

Lemma 75. *If S_u, S_v are connected to S_w and $d(S_u, S_w) - d(S_v, S_w) \geq 2$, then $S_u \cap S_v = \emptyset$.*

Proof. Assume otherwise that $S_u \cap S_v \neq \emptyset$. Then, by Lemma 43, either S_u hits S_v or S_v hits S_u or $b_u = b_v$. If S_u hits S_v, then $d(S_u, S_w) \leq d(S_v, S_w) + 1$, a contradiction. If S_v hits S_u, then by Lemma 67, $d(S_u, S_w) \leq d(S_v, S_w)$. Note that if S_v has distance 0, then S_u cannot intersect S_v, by Lemma 60. If $b_u = b_v$, then $d(S_u, S_w) = d(S_v, S_w)$, a contradiction. ☐

Lemma 76. *If $S_u \subseteq C_v \subsetneq C_w$, then $d(S_u, S_v) < d(S_u, S_w)$.*

Proof. Note $C_v \subsetneq C_w$ implies that $d(S_v, S_w) > 0$, or else S_v, S_w would share the same vertex and $C_v = C_w$. So, S_u has a connection of length $L = d(S_u, S_w) > 0$ to S_w, and there exists $S_{v_1}, ..., S_{v_{L+1}}$ such that $S_u = S_{v_1}$ and S_w shares its bad vertex with $S_{v_{L+1}}$ and for $1 \leq i \leq L$, S_{v_i} shares its bad vertex with an elimination set S_{w_i} such that S_{w_i} hits $S_{v_{i+1}}$.

But by Lemma 66, the only out-neighbors of C_v is from b_v and since $S_w \nsubseteq C_v$, there must exists $k \leq L$ such that $b_v \in S_{v_k}$. So, S_u is connected to S_{v_k} with smaller length and S_{v_k} is connected to S_v with length 0, so $d(S_u, S_v) < d(S_u, S_w)$. ☐

Lemma 77. *M_u is the set of all minimally connected sets to S_u.*

Proof. Let S_v be minimally connected to S_u. If it is in $C_w \subsetneq C_u$, then by Lemma 76, we have a contradiction of minimality. Therefore, all minimally connected sets to S_u are in M_u.

Let $S_u \subseteq M_u$, then note that it must be connected to S_u. If it is not minimally connected to S_u and is minimally connected to another elimination set S_w, then $C_w \subsetneq C_u$. But, this contradicts $S_u \subseteq M_u$, so M_u is in the set of all minimally sets to S_u. ☐

Therefore, we note that if $S_w \subseteq M_u$ is a non A-type, then $d(S_w, S_u) = 0$. And, we see that $b_w = b_u$ is implied. We now consider an equivalence relation on all minimally connected sets $M_{v_1}, ..., M_{v_r}$ using set equivalence avoiding duplicates. The choice of the representatives of these equivalence classes is arbitrary.

Under the equivalence relation quotient, by Lemma 71 and Lemma 77, note that none of the minimally connected sets intersect. Furthermore, it is easy to see that if $(u, v) \in E(G)$ and $u \in M_w, v \in M_w^c$, then $u = b_w$, as for connected sets.

8 Finiteness of Minimal Forbidden Minors

8.1 Finiteness of Minimally Connected Sets

Definition 78. Let M_u be a minimally connected set. Define the **extra set** of M_u, $T_u \subseteq S_0$, to be the smallest subset of S_0 such that there does not exist $v \in S_0 \cap T_u^c$ such that $N_{out}(v) \subseteq M_u \cup T_u$.

Note that this is a recursive definition of T_u and T_u is not the maximal/maximum set in S_0 that hits M_u. However, it is true that T_u hits M_u. If there are no vertices in S_0 with all of its out-neighbors in M_u, then $T_u = \emptyset$. Since T_u hits M_u, we have that if $(w, v) \in E(G)$ and $w \in M_u \cup T_u, v \in (M_u \cup T_u)^c$, then $w = b_u$, as before.

Lemma 79. $(M_u \cup T_u) \cap (M_v \cup T_v) = \emptyset$ *for* $u \neq v$.

Proof. Let $H = (M_u \cup T_u) \cap (M_v \cup T_v)$. If H is non-empty, then it must have out-neighbors in H^c. So, by our above observation, either $b_u \in M_v \cup T_v$ or $b_v \in M_u \cup T_u$. But since $M_u \cap M_v = \emptyset$, we either have $b_u \in T_v$ or $b_v \in T_u$. However, $T_u, T_v \subseteq S_0$, a contradiction. □

Lemma 80. *Let* S_u *be an non A-type and* H *be the set of all elimination sets that share the same bad vertex as* S_u. *Then,* S_u *has at most 2 out-neighbors in* H^c.

Proof. Assume otherwise and let $w_1, w_2, w_3 \in H^c$ be three out-neighbors of S_u. Consider deleting (b_u, w_1) to obtain G'. Note that G' might not be strongly connected. If so, we can find $A, B \subseteq V(G')$ such that $w_1 \in A, b_u \in B$ and all edges go from A to B. Note that if $v \in B$, then $N_{out}(v) \subseteq B$.

Now, consider $G'' = G'[B]$, which is strongly connected. If $S_{u_1}, ..., S_{u_s}$ is the elimination decomposition of G, then we claim that we can appeal to the characterization theorem with $V_0 = S_0 \cap B$ and $V_j = S_{u_j} \cap B$. It is clear that $\bigcup_{i=0}^{s} V_i = V(G'')$.

Next, we claim that $V_j \neq G''$, $j \geq 1$. If $V_j = G''$, then $b_u \in V_j$, and so $b_u \in S_{u_j}$ in G. If $b_{u_j} \neq b_u$, then S_u is an A-type, a contradiction, so we must have $b_u = b_{u_j}$. But, this means that there are at least two out-neighbors from b_{u_j} to $S_{u_j}^c$ in G and in G', we have only deleted one edge, so there must still be an out-neighbor of b_{u_j} in $S_{u_j}^c$. Note that the out-neighbor is in B, since it is an out-neighbor of $b_u \in B$. Therefore, it is in $G'' \cap S_{u_j}^c$, contradicting $V_j = G''$. We conclude that V_j are still strict subsets.

We see that if $v \in S_0$, then v still has outdegree ≥ 2 in G'. If $v \in G''$, then its outdegree is not reduced, by our previous observation. Thus, V_0 is a set of vertices with outdegree ≥ 2.

Now, if V_j does not contain b_{u_j}, then $V_j = \emptyset$ since otherwise V_j has no out-neighbors in $G'' \cap V_j^c$. For property 3, note that if $b_{u_j} \neq b_u$ and $b_{u_j} \in B$, then V_j still has at least two out-neighbors to V_j^c as $N_{out}(b_{u_j}) \subseteq B$. If $b_u = b_{u_j}$, then $u_2, u_3 \notin V_j$ since $u_2, u_3 \notin H$ and $S_{u_j} \subseteq H$. So property 3 holds.

Now, for property 4, if $w_j \in V_j^c$ and $x_j \in V_j$ such that $(w_j, x_j) \in E(G'')$, then we see that in G, there existed $y_j \in V_j^c$ such that $(w_j, y_j) \in E(G)$. Note that if $(w_j, y_j) \in E(G')$, then since $w_j \in B, y_j \in B$, so $(w_j, y_j) \in E(G'')$. This does not hold only if $w_j = b_u$ and $y_j = u_1$. Note that if u_2 or u_3 is not in V_j, then property 4 still holds. We only run into a problem if we can find an elimination set S_{v_1} such that $u_1 \in S_{v_1}^c$ and $u_2, u_3 \in S_{v_1}$.

Now, consider deleting $(b_u, u_2), (b_u, u_3)$, and we can find S_{v_2}, S_{v_3} so that $u_2 \notin S_{v_2}$, $u_1, u_3 \in S_{v_2}$ and $u_3 \notin S_{v_3}$, $u_1, u_2 \in S_{v_3}$. It is clear that $S_{v_1}, S_{v_2}, S_{v_3}$ mutually intersect. We see that $S_{v_1} \cap S_{v_2} \cap S_{v_3} \neq \emptyset$ or else, by Lemma 46, they would form 3 A-types that hit each other, contradicting the existence of S_u. But $u_1 \in S_{v_1}^c \cap S_{v_2} \cap S_{v_3}$ and $u_2 \in S_{v_1} \cap S_{v_2}^c \cap S_{v_3}$ contradicts Lemma 57.

So property 4 holds, and G'' is a minor of G with directed pathwidth ≥ 2, contradicting the minimality of G. □

Corollary 81. *Let* M_u *be a minimally connected set.* M_u *has at most 2 out-neighbors to* M_u^c.

Lemma 82. *If* M_u *has at least 2 out-neighbors in* $(M_u \cup T_u)^c$, *then there exists another* M_v *that has at least 2 out-neighbors in* $M_u \cup T_u$.

Proof. Consider reducing u in M_u to get M'_u and we claim that our resulting graph G' is still of directed pathwidth 2, unless we can find a minimally connected set M_v with at least two out-neighbors in $M_u \cup T_u$.

Assume for the sake of contradiction that such a M_v does not exist. Let $S_{u_1}, ..., S_{u_s}$ be the elimination decomposition of G. We try to apply characterization theorem with $V_0 = S_0 \cap T_u^c$ and $V_j = S_{u_j} \cap M_u^c$, for $1 \leq j \leq s$. Lastly, we let $V_{s+1} = M'_u \cup T_u$. Clearly, $\bigcup_{i=0}^{s+1} V_i = V(G')$.

Note that all V_j are strict subsets as before and V_{s+1} is a strict subset since $M_u \cup T_u$ has out-neighbors in $(M_u \cup T_u)^c$ in G and thus, V_{s+1} is a strict subset in G'. Furthermore, note that S_0 still have vertices of outdegree ≥ 2 in G'.

Let w be the out-neighbor of u. For property 3, for $1 \leq j \leq s$, if $V_j = S_{u_j} \cap M_u^c$ is non-empty and has lost out-neighbors in V_j^c after the contraction of u, then V_j must have u, w as its out-neighbors in G. We see that $V_j \subseteq M_u^c$ and so there must exist some M_v such that $V_j \subseteq M_v$. Clearly, M_v has at least two out-neighbors in $M_u \cup T_u$ and this contradicts our assumption that no such M_v exists. So, we see that property 3 holds. For V_{s+1}, property 3 holds by assumption.

For property 4, for $1 \leq j \leq s$, realize in G, if $x \in S_{u_j}^c$ and there exists $y \in S_{u_j}$ such that $(x, y) \in E(G)$, then there exists $z \in S_{u_j}^c$ such that $(x, z) \in E(G)$. When $V_j = S_{u_j} \cap M_u^c$ is non-empty, note that unless we have $x = w$ and $z = u$, $(x, z) \in E(G')$ holds or even if $z = u$, then $(x, w) \in E(G')$ and $w \in S_{u_j}^c$ holds. Since $V_j \subseteq M_u^c$, it follows that $x = w = b_u$.

If b_u has all of its out-neighbors in S_{u_j} in G', then we see that in G, $N_{out}(b_u) \subseteq S_{u_j} \cup \{u\}$. This implies that S_u hits S_{u_j}, contradicting that S_u is a non A-type.

For $V_{s+1} = M'_u \cup T_u$, and if $x \in V_{s+1}^c$ and there exists $y \in V_{s+1}$ such that $(x, y) \in E(G)$, then we consider the following cases.

Case 1: If $x \in S_0$, then we can find $z \in V_{s+1}^c$ such that $(x, z) \in E(G')$ unless and $N_{out}(x) \subseteq V_{s+1}$. However, this implies that $x \in T_u$, a contradiction.

Case 2: If $x \in M_v$, then $x = b_v$. Note that in G, we can find $z \neq x$ such that $(x, z) \in E(G)$. If $z \notin M_u \cup T_u$, then $z \notin V_{s+1}$ and $(x, z) \in E(G')$, so property 4 holds. Otherwise, $z \in M_u \cup T_u$ and M_v has at least two out-neighbors $y, z \in V_{s+1}$, but this contradicts our assumption.

Now, by the characterization theorem, G' having pathwidth ≥ 2 contradicts the minimality of G. Therefore, we must be able to find M_v such that M_v has at least two out-neighbors in $M_u \cup T_u$. □

Lemma 83. *If M_u has at least 2 out-neighbors in $(M_u \cup T_u)^c$, then there are exactly two minimal connected sets and its extra set and they hit each other.*

Proof. By Lemma 82, we can find M_{u_1} such that it has at least two out-neighbors in $M_u \cup T_u$. By Lemma 79, $M_u \cup T_u \subseteq (M_{u_1} \cup T_{u_1})^c$ and by Lemma 80, $M_u \cup T_u$ has exactly 2 out-neighbors in $M_u \cup T_u$. In fact, the outdegree restriction implies $M_{u_1} \cup T_{u_1}$ hits $M_u \cup T_u$. And appealing to 82 again, we can find M_{u_2} such that it has at least two out-neighbors in $M_{u_1} \cup T_{u_1}$.

Let $M_u = M_{u_0}$. We can keep doing this to find a sequence of $M_{u_1}, M_{u_2}, ...$ and since the graph is finite, there must exist $0 \leq m < n$ such that $M_{u_m} = M_{u_n}$. We recall that $M_{u_i} \cup T_{u_i}$ hits $M_{u_{i+1}} \cup T_{u_{i+1}}$ in this sequence for $i \geq 0$. So, $H = \bigcup_{k=m}^{n-1} (M_{u_k} \cup T_{u_k})$ has no out-neighbors in H^c.

So, our graph G is made up t different minimally connected sets and their extra sets that hit each other in a cycle. Assume that $t > 2$ and relabel our minimally connected sets to be $M_{u_1}, ..., M_{u_t}$. Let $S_{v_1}, ..., S_{v_s}$ be the elimination decomposition of G.

Consider the contraction u_2 in M_{u_2} to get M'_{u_2}. Now, we appeal to characterization theorem in the resulting graph G' with $V_0 = S_0 \cap T_{u_2}^c$ and for $1 \leq j \leq s$

$$V_j = \begin{cases} S_{v_j} \cap (M'_{u_2})^c & b_{v_j} \neq b_{u_1} \\ S_{v_j} \cup M'_{u_2} \cup T_{u_2} & b_{v_j} = b_{u_1} \end{cases}$$

It is clear that $\bigcup_{i=0}^{s} V_i = V(G')$. Note that these sets V_j are all strict subsets of $V(G')$ since each S_{v_j} is in one of $M_{u_1}, ..., M_{u_t}$ and $t > 2$, so there must exists some M_v such that $V_j \not\subseteq M_v$.

For property 3, for $1 \leq j \leq s$, if $b_{v_j} \neq b_{u_1}$, then V_j does not have out-neighbors in $M_{u_2} \cup T_{u_2}$, so $V_j = S_{v_j} \cap M'_{u_2})^c$ still has two out-neighbors in V_j^c. If $b_{v_j} = b_{u_1}$, then $V_j = S_{v_j} \cup M'_{u_2} \cup T_{u_2}$ and note that b_{u_2} has at least two out-neighbors in $M_{u_3} \cup T_{u_3}$. Therefore, property 3 holds.

For property 4, for $1 \leq j \leq s$, if $b_{v_j} \neq b_{u_1}$, then if $(x, y) \in E(G')$ with $x \in V_j^c, y \in V_j$, then we can find $z \in S_{v_j}^c$ such that $(x, z) \in E(G)$. Note that $(x, z) \in E(G')$ unless $z = u_2$. Let w_2 be the out-neighbor of u_2. If $x \neq w_2$, then $(x, w_2) \in E(G')$ and $w_2 \in V_j^c$ since $S_{v_j} \subseteq M_{u_2}^c$. If $x = w_2, z = u_2$, then $x = w_2 = b_{u_2}$ is forced. But, $N_{out}(u_2) \subseteq S_{v_j} \cup \{u_2\}$ implies that S_{u_2} hits S_{v_j}, contradicting that S_{u_2} is a non A-type. So, we can find $z \in V_j^c$ with $(x, z) \in E(G')$.

If $b_{v_j} = b_{u_1}$, then we claim that $y \in S_{v_j}$. If $y \notin S_{v_j}$, then $y \in M'_{u_2} \cup T_{u_2}$. But since only $M_{u_1} \cup T_{u_1}$ has out-neighbors in $M_{u_2} \cup T_{u_2}$, we must have $x \in M_{u_1} \cup T_{u_1}$, implying $x = b_{u_1}$, a contradiction since $x \in V_j$. Therefore, $y \in S_{v_j}$ and we can find $z \in S_{v_j}^c$ such that $(x, z) \in E(G)$. Note that $z \notin M_{u_2} \cup T_{u_2}$ since $x \in M_{u_1}$ forces $x = b_{u_1}$, leading to the same contradiction. Therefore, $z \notin V_j$ and $(x, z) \in E(G')$.

So, G' has directed pathwidth ≥ 2, contradicting the minimality of G. So, $t \leq 2$, but we see that $t \geq 2$ since by Lemma 82, there must exist another $M_v \cup T_v$ that hits $M_u \cup T_u$. So, $t = 2$. ☐

Theorem 84. *There are at most 2 minimally connected sets.*

Proof. If there exists M_u such that $M_u \cup T_u$ has at least 2 out-neighbors in $(M_u \cup T_u)^c$, then by Lemma 83, we have exactly 2 minimally connected set. So, all minimally connected sets M_u has at most 1 out-neighbor in $(M_u \cup T_u)^c$.

Assume that there are more than 2 minimally connected sets. Let M_u, M_v be two different such sets. By strong connectivity, there is a directed path from b_u to M_v, denoted as $P(b_u, M_v)$. We see that except for the endpoints, the vertices of $P(b_u, M_v)$ are in $(M_u \cup T_u \cup M_v \cup T_v)^c$. Similarly, we can find a reverse path $P(b_v, M_u)$ with such properties.

If $x, y \in P(b_v, M_u) \cap P(b_u, M_v)$, then note that either there is a directed cycle that includes x, y and we can cycle contract, or if not, then $x \neq b_u, b_v$, so we may just out-contract. By cycle contraction, it worth noting that b_u is now reduced to b_v. So, we may assume that the paths share at most one common vertex.

Otherwise, out-contract each vertex in $(M_u \cup T_u \cup M_v \cup T_v)^c$ until each path at most 2 edges (one from b_u to the common vertex and the common vertex to M_v). Using the characterization theorem, we claim that this subgraph is still of pathwidth ≥ 2.

Consider the new graph obtained and let $H = M_u \cup T_u \cup M_v \cup T_v \cup \{c\}$, where c is the possibly non-existent common vertex. Now, let $S_{v_1}, ..., S_{v_s}$ be the elimination sets contained in M_u, M_v and let $V_0 = T_u \cup T_v \cup \{c\}$, $V_j = S_{v_j}$ for $1 \leq j \leq s$. Clearly, $\bigcup_{i=0}^{s} V_j = V(H)$ and all V_j are strict subsets of H.

Next, note that T_u, T_v has no out-neighbors in $(M_u \cup M_v)^c$ in G, so they are still of out-degree ≥ 2. If c exists, then note that is has one out-neighbor in M_u and one in M_v. So, V_0 has vertices of out-degree ≥ 2.

It is important to observe that in G, b_u has at most 1 out-neighbor in $(M_u \cup T_u)^c$, meaning that b_u has an out-neighbor $w \in V_j^c \cap (M_u \cup T_u)$. In H, note that b_u also has an out-neighbor in $(M_u \cup T_u)^c$ and so, it has two distinct vertices in V_j^c.

For property 3, if $V_j = S_{v_j}$ does not satisfy $b_{v_j} = b_u$ or $b_{v_j} = b_v$, then $N_{out}(S_{v_j})$ is strictly contained in M_u, M_v, so b_{v_j} still has two out-neighbors in $S_{v_j}^c$. If $b_{v_j} = b_u$, then our observation above gives us property 3. Similarly if $b_{v_j} = b_v$.

For property 4, if $(x, y) \in E(H)$ and $x \in V_j, y \in V_j^c$, then note that in G, there exists $z \in V_j^c$ such that $(x, z) \in E(G)$. If $x \neq b_u, b_v$, then $z \in H$ and $(x, z) \in E(H)$. Otherwise, $x = b_u$ and if $V_j \subseteq M_u \cup T_u$, then in H, b_u has an out-neighbor in $(M_u \cup T_u)^c$ by the above observation. If $V_j \subseteq M_v \cup T_v$, then again, b_u has an out-neighbor in $M_u \cup T_u \subseteq (M_v \cup T_v)^c$. Similarly if $x = b_v$.

So, by the characterization theorem, H contradicts the minimality of G and there are at most 2 minimally connected sets. □

8.2 Finiteness of Elimination Sets

In previous section we gave a bound on the number of minimally connected sets. Furthermore, we have bound the number of out-neighbors of a minimally connected set (and its extra set) to its complement to be at most 2. From this, we intuit that a minimally connected set cannot contain many elimination sets and these elimination sets cannot have large distance.

In this section, we show that the number of elimination sets is indeed finite. We will show that for any minimally connected set M_u, the number of elimination sets in M_u is at most 4. To understand the behavior of M_u, we start by understanding the behavior of S_0 and the extra sets.

Theorem 85. *All vertices in S_0 have outdegree exactly 2.*

Proof. Let $S_{u_1}, ..., S_{u_s}$ be the elimination decomposition of G and assume that $v \in S_0$ have more than 2 out-neighbors, say w_1, w_2, w_3. Consider deleting (v, w_1) to get G'. Note that G' might not be strongly connected. If so, we can find $A, B \subseteq V(G')$ such that $w_1 \in A, v \in B$ and all edges go from A to B. Note that if $v \in B$, then $N_{out}(v) \subseteq B$.

Now, consider $G'' = G'[B]$, which is strongly connected. Consider appealing to the characterization theorem with $V_0 = S_0 \cap B$ and $V_j = S_{u_j} \cap B$ for $1 \leq j \leq s$. Clearly, we have $\bigcup_{j=0}^{s} V_j = V(G'')$. Furthermore, for $1 \leq j \leq s$, V_j is a strict subset of G'' since $v \in G''$ and $v \notin V_j$. Also, V_0 has vertices of outdegree ≥ 2 since all vertices in V_0, including v, still retained outdegree ≥ 2 in G'. Then, in G'', those vertices in B did not lose outdegree, so V_0 still has vertices of outdegree ≥ 2.

For property 3, if $V_j \neq \emptyset$, note that by Lemma 33, $b_{u_j} \in V_j$. Now, b_{u_j} is still the only vertex in V_j with out-neighbors in V_j^c and all its out-neighbors are in G''. For property 4, we want to show that if $(x, y) \in E(G'')$ and $x \in V_j^c, y \in V_j$, then there exists $z \in V_j^c$ such that $(x, z) \in E(G'')$. Note that in G, we can find $z \in S_{v_j}^c$ such that $(x, z) \in E(G)$. Unless $x = v, z = w_1$, $z \in B$ since $x \in B$ and $(x, z) \in E(G'')$.

So, we only have a problem when there exists S_{v_1} such that $N_{out}(v) \subseteq S_{v_1} \cup \{w_1\}$ and $w_1 \notin S_{v_1}$. Now, consider deleting $(v, w_2), (v, w_3)$ and we can similarly find S_{v_2}, S_{v_3} such that $N_{out}(v) \subseteq S_{v_2} \cup \{w_2\}$ and $w_2 \notin S_{v_2}$, $N_{out}(v) \subseteq S_{v_3} \cup \{w_3\}$ and $w_3 \notin S_{v_3}$.

But, $w_1 \in S_{v_1}^c \cap S_{v_2} \cap S_{v_3}$, $w_2 \in S_{v_1} \cap S_{v_2}^c \cap S_{v_3}$, $w_3 \in S_{v_1} \cap S_{v_2} \cap S_{v_3}^c$. Since $S_{v_1}, S_{v_2}, S_{v_3}$ pairwise intersect, by Lemma 46, we must have $S_{v_1} \cap S_{v_2} \cap S_{v_3} \neq \emptyset$, or else the existence of v would be contradicted. But, w_1, w_2 contradicts Lemma 57.

We conclude that property 4 must hold for one of w_1, w_2, w_3 and deleting the respective edge gives G' with directed pathwidth ≥ 2, by the characterization theorem. This contradicts the minimality of G. □

Lemma 86. *Let $u, v \in S_0$ and $(u, v) \in E(G)$ and let G' be graph obtained by out-contracting (u, v). Then, if $S'_0 = S_0 \setminus \{u\}$ has vertices of outdegree ≥ 2 in G' and there does not exist $w \in V(G)$ such that w has u, v as out-neighbors, then G' has directed pathwidth ≥ 2.*

Proof. Let $S_{u_1}, ..., S_{u_s}$ be the elimination decomposition of G and let G' be defined as above. Note that G' might not be strongly connected, in which case, we can find $A, B \subseteq V(G')$ such that $v \in B$.

Now, consider $G'' = G'[B]$ and $V_0 = S'_0 \cap B$ and $V_j = S_{u_j} \cap B$. Note that all the properties hold. The first assumption gives property 2 and the second assumption gives property 3,4. By the same argument in Theorem 85, the characterization theorem gives us that G'' has directed pathwidth ≥ 2. This also implies that G' has directed pathwidth ≥ 2. □

Theorem 87. *For any M_u, $|T_u| \leq 3$. There exists only one vertex $v \in T_u$ such that $N_{out}(v) \cap M_u \neq \emptyset$.*

Proof. It is crucial to note that there are no directed cycles in $G[T_u]$, which can be quickly seen from the definition. This is because if $C \subseteq T_u$ has a directed cycle, then let $H \subseteq T_u$ be all the vertices in H with a directed path in $G[T_u]$ to some vertex in C. Consider $T'_u = T_u \setminus H$ and notice that T'_u has out-neighbors only in $T'_u \cup M_u$ since no vertex in T'_u has out-neighbors in H. Furthermore, if $v \in H$, then v has an out-neighbor in H. Therefore, T'_u contradicts the minimality of T_u.

Case 1: Assume that we can find a M_u in G such $M_u \cup T_u$ has 2 out-neighbors in $(M_u \cup T_u)^c$. In this case, by Lemma 83, the graph only contains another $M_v \cup T_v$ such that $M_u \cup T_u$ and $M_v \cup T_v$ hit each other. By Lemma 80, M_u has no out-neighbors in T_u and similarly for M_v.

If M_v does not have out-neighbors in T_u, then $T_u = \emptyset$ by strongly connectivity. Otherwise, M_v has out-neighbors in T_u and since $G[T_u]$ is a directed acyclic graph, let $w \in T_u$ be a vertex with no in-neighbors in T_u and some out-neighbor in T_u. Let $x \in T_u$ be an out-neighbor of w in T_u.

Consider out-contracting (w, x) in T_u to get T'_u. Note that $T_u \setminus \{w\}$ has no out-neighbors to w, so only M_v (or b_v) has w as an out-neighbor. Therefore, T'_u still has vertices of outdegree ≥ 2. Consider out-contracting w to its out-neighbor in T_u and since T'_u still has vertices of outdegree ≥ 2. If b_v does not have w, x as out-neighbors, then Lemma 86 gives a contradiction to the minimality of G. We immediately deduce that if M_v only has one out-neighbor in T_u, then $|T_u| \leq 1$.

Otherwise, M_v has two out-neighbors in T_u, then w, x must be its only out-neighbors in T_u. Note that x has a directed path P_{xy} to $y \in T_u$, where $N_{out}(y) \subseteq M_v$ due to the DAG property. By Theorem 85, we see that w has one other out-neighbor z.

If z is on the path, then consider out-contracting P_{xz} until P_{xz} becomes only an edge (x, z). Then, out-contract P_{zy} until z, y are identified. out-contraction preserves outdegree of w to be ≥ 2 since $G[T_u]$ is a directed acyclic graph. Delete any other vertices in T_u. Note that w, x, z has outdegree ≥ 2 and by the characterization theorem, G still has directed pathwidth ≥ 2. Thus, $|T_u| \leq 3$.

Note that x could have out-neighbors in M_u. If x does, then let y be a be its out-neighbor in M_u. First, out-contract (x, a) and next out-contract (w, z) and note that z is the only remaining vertex in T_u and $|T_u| \leq 1$.

If z is not on the path and there is a path P_{za}, where $a \in P_{xy}$, then first out-contract P_{za}, and then follow the same argument above. Otherwise, there is a path P_{za} that does not intersect P_{xy} such that $a \in M_u$. Then fully out-contract P_{xy} until x, y are identified. Then, fully out-contract P_{za} until z and a are identified. Then, delete anything in T_u until only $x = y$ is left in T_u and it has outdegree ≥ 2. By the characterization theorem, it can be checked that G still has directed pathwidth ≥ 2. Thus, $|T_u| \leq 1$.

Case 2: All $M_u \cup T_u$ only has at most 1 out-neighbor in $(M_u \cup T_u)^c$. If there exists a $M_u \cup T_u$ with exactly 1 out-neighbor in $(M_u \cup T_u)^c$, then by Lemma 84, G has at most one other $M_v \cup T_v$ with also exactly 1 out-neighbor in $(M_u \cup T_u)^c$.

Case 2.1: If M_v exists, note that M_u has at most 1 out-neighbor in T_u, by Lemma 80, and M_v also has at most 1 out-neighbor in T_u. We conclude that there does not exist w with two out-neighbors in T_u and so, the same argument in case 1 holds and we deduce that $|T_u| \leq 1$.

Case 2.2: If M_v does not exist, then note that $H = (M_u \cup T_u)^c$ must be a subset of S_0. But by Theorem 85, if $w \in H$, then w has outdegree ≥ 2 and since $w \notin T_u$, w has an out-neighbor in H. We conclude that w does not have two out-neighbors in T_u and so, the same argument in case 1 holds and we deduce that $|T_u| \leq 1$.

Case 2.3: $M_u \cup T_u$ has no out-neighbors in $(M_u \cup T_u)^c$. In this case, M_u has both its out-neighbors in T_u and the same argument in case 1 holds and we deduce that $|T_u| \leq 3$. ☐

Lemma 88. *If $b_u, b_v, b_w \in S_u \cap S_v \cap S_w$ and b_u has out-neighbors in $S_u \cap S_v^c \cap S_w^c$ and $S_u^c \cap S_v \cap S_w^c$, then S_u, S_v, S_w are the only elimination sets in G.*

Proof. Consider $H = S_w \cap S_u^c \cap S_v^c$, which is not empty, since $v_k \in H$. By strong connectivity, b_u has a directed path to H. Note that if the path is completely contained in $I = S_u \cup S_v \cup S_w$, then there exists $x \in I \cap H^c, y \in H$ such that $(x, y) \in E(G)$. Since $x \in S_u \cup S_v$ and $y \in S_u^c \cap S_v^c$, it follows that $x = b_u$. Then, I is a minimal minor by using the characterization theorem on S_u, S_v, S_w.

Otherwise, the simple path from b_u to H is in I^c, except for the endpoints. We can out-contract all vertices in I^c along the path and deduce the resulting graph to have directed pathwidth ≥ 2. ☐

Lemma 89. *Let $G \in \mathcal{F}$. If there are at least two bad vertices in G or $S_0 \neq \emptyset$, then there cannot exist S_u such that b_u has more than 1 out-neighbor in S_u.*

Proof. Let G be a minimal forbidden minor with at least 2 bad vertices or $S_0 \neq \emptyset$ and assume for contradiction there exists S_u be a maximal elimination set such that b_u has at least 2 out-neighbors in S_u, say v, w. Let H be the vertex set that contains all S_x such that $b_x = b_u$. Since there are at least 2 bad vertices or $S_0 \neq \emptyset$, we know that H is a strict subset of G and by strong connectivity, it must have an out-neighbor in H^c. So, by Lemma 33, it must be b_u that has an out-neighbor in G/H, say h.

Let $S_{v_1}, ..., S_{v_s}$ be the elimination decomposition of G. For now, consider deleting (b_u, w) to get G'. Note that deleting (b_u, w) might have removed strong connectivity, so in G', there might be vertices in S_u that cannot be reached from b_u. Note that all other vertices in S_u^c can still be reached from b_u in G' since $b_u \in G'$.

So, we can find two subsets A, B such that all edges go from A to B and $w \in A, A \subseteq S_u, b_u \in B$. Now, consider the graph $G'' = G'[B]$ and invoking the characterization theorem with $V_0 = S_0 \cap B, V_i = S_{v_i} \cap B$ for $1 \leq i \leq s$. Note that removing A does not reduce outdegree from G' to G''. And from G to G'', only the outdegree of b_u, which was reduced by 1. So, V_0 still is a set of vertices of outdegree ≥ 2. And since A is a strict subset of S_u, we see that we have $V_0, V_1, ..., V_n$ are still strict subsets of G'' with $\bigcup_{i=0}^{s} V_i = V(G'')$.

Now, for property 3 to hold, we only run into a problem if there exists S_x with $b_x = b_u$ that contains all out-neighbors of b_u except w, h.

Case 1: Assume such a S_x exists. Then, consider deleting (b_u, v) instead and note that we run into a problem if there exists S_y with $b_y = b_u$ that contains all but v, h. However, we see that $b_y \in S_u \cap S_x \cap S_y$, but $v \in S_u \cap S_x \cap S_y^c$ and $w \in S_u \cap S_x^c \cap S_y$, a contradiction to Lemma 57

Now, consider property 4, and note that it holds unless there exists S_y that contains all out-neighbors of b_u except v. Note that S_u, S_v, S_w mutually intersect and therefore, by Lemma 46, they must have a triple intersection. However, this violates Lemma 57 as before. So property 4 must hold.

Case 2: Such a S_x does not exist. By symmetry, we may assume there does not exist S_y with $b_y = b_u$ that contains all out-neighbors except v, h. So, property 3 holds if we delete (b_u, w) or (b_u, v).

So, consider deleting (b_u, w) again, and note that property 4 holds unless we can find some S_y that contains all out-neighbors of b_u except w. Then, delete (b_u, v) and that fails only if S_z contains all out-neighbors of b_u except v. But again, S_u, S_y, S_z mutually intersect, and as before, we derive a contradiction.

Therefore, deleting one of $(b_u, v), (b_u, w)$ gives G', which is still of directed pathwidth ≥ 2, a contradiction.

□

Lemma 90. *Let M_u be a minimally connected set with more than 4 elimination sets. For any $S_v \subseteq M_u$, there exists S_w such that either S_v hits S_w or S_w hits S_v or $b_v = b_w$. Furthermore, either*

1) There exists a non A-type S_x such that b_x has out-neighbors in $S_v \cap S_w^c$ and $S_v^c \cap S_w$ and $N_{out}(b_x) \subseteq S_v \cup S_w$

2) There exists $x \in S_0$ such that such that x has out-neighbors in $S_v \cap S_w^c$ and $S_v^c \cap S_w$ and $N_{out}(x) \subseteq S_v \cup S_w$

3) $b_v = b_u$ and b_u has an out-neighbor in $S_w \cap S_v^c$ and b_u has only 1 out-neighbor in $(S_w \cup S_v)^c$

4) $b_w = b_u$ and b_u has an out-neighbor in $S_v \cap S_w^c$ and b_u has only 1 out-neighbor in $(S_w \cup S_v)^c$

Proof. By Lemma 84, we can have at most two minimally connected sets. Furthermore, it is important to note that by Lemma 82, if S_v is any elimination set in M_u, then b_v has at most 2 out-neighbors in M_u^c.

Now, consider any $S_v \subseteq M_u$ and reducing v to get S_v'. By minimality, there must exists an elimination ordering $u_1, ..., u_m$ and its corresponding out-sequence such that $R(G, [u_1, ..., u_m])$ is only 1 vertex. Realize $u_1 \notin S_0$ or else $N_{out}(u_1) \subseteq S_v$ follows. Also, $u_1 \notin S_v$, or else it is clear that we cannot eliminate any vertex in $S_v'^c$. Thus, u_1 is in some elimination set $S_w \neq S_v$. We may assume that $u_1 = w$.

We claim that either S_v hits S_w or S_w hits S_v or $b_v = b_w$. If $S_v \cap S_w \neq \emptyset$, then we are done by Lemma 41 and Lemma 43. So, we may assume that $S_v \cap S_w = \emptyset$. Let p be minimal such that $u_p \notin S_w$ or $v_p \notin S_w$.

Case 1: $u_p \notin S_w$. We claim that $u_p \in S_v$. Since u_p can be contracted in G', we see that in G', $N_{out}(u_p) \subseteq S_w$. However, $N_{out}(u_p) \subsetneq S_w$, so it must be that u_p has v as an out-neighbor. Let x be the out-neighbor of v. However, if $u_p \notin S_v$, then u_p would still have x as an out-neighbor in G' and $x \notin S_w$ since x is lonely. We conclude that $u_p \in S_v$.

Note that since S_v, S_w do not intersect, p is minimal such that $u_p \in S_v$ and since u_p has out-neighbors in S_v^c, $u_p = b_v$ is forced. This implies that $N_{out}(u_p) \subseteq S_v \cup S_w$, implies that S_v hits S_w.

Case 2: $v_p \notin S_w$. Similarly, we claim that $v_p \in S_v$. Note that outdegree of u_p must have been reduced in G', so u_p must have x as an out-neighbor in G'. Since $x \notin S_w$, it follows that $v_p = x \in S_v$. Then, since b_w has not been reduced, we conclude that $u_p = b_w$. So, $N_{out}(u_p) \subseteq S_v \cup S_w$, implying that S_w hits S_v.

Now, consider the minimal n such that either $u_n \notin S_v \cup S_w$ or $v_n \notin S_v \cup S_w$.

Case 1: $u_n \notin S_v \cup S_w$. In G, we see that $N_{out}(u_n) \subseteq S_v \cup S_w$ and by Lemma 34, u_n has an out-neighbor in $w_1 \in S_v \cap S_w^c$ and $w_2 \in S_v^c \cap S_w$. We claim that $u_n \in S_0$ or that it is a bad vertex of a non A-type. If not, then either u_n is a bad vertex of an A-type or u_n is contained in some elimination set S_y but is not the bad vertex of S_y.

Case 1.1: u_n is a bad vertex of an A-type S_x and let S_x hits S_y. Since $N_{out}(u_n) \subseteq S_x \cup S_y$, we see that $w_1, w_2 \in S_x \cup S_y$. If $w_1, w_2 \in S_x$, then $w_1 \in S_v \cap S_w^c \cap S_x$ and $w_2 \in S_v^c \cap S_w \cap S_x$, contradicting Lemma 57. Similarly for S_y. Without loss of generality, let $w_1 \in S_x \cap S_y^c$ and $w_2 \in S_y \cap S_x^c$. Note that $u_n = b_x \notin S_v$, but $w_1 \in S_x \cap S_v$, so S_v hits S_x and $b_v \in S_x$. From above, we now split into cases: S_v hits S_w, S_w hits S_v, or $b_v = b_w$.

Case 1.1.1: S_v hits S_w. Then, $S_w \cap S_x \neq \emptyset$ and since $b_x \notin S_w$, we conclude that S_w hits S_x and $b_w \in S_x$. Note that S_y cannot hit S_w, or else S_x, S_y, S_w hits each other in a cycle. So, $b_w \in S_y$. And, notice $b_w \in S_x \cap S_y \cap S_w$. Since S_y cannot hit S_x, it also follows that $b_x \in S_y$. But, $b_x \in S_x \cap S_y \cap S_w^c$ and $w_2 \in S_x^c \cap S_y \cap S_w$, contradicting Lemma 57.

Case 1.1.2: S_w hits S_v. If S_y hits S_w, then we have a cycle of A-types that hit each other in a cycle, so $b_w \in S_y$. Now, either $b_w = b_y$ or $b_w \neq b_y$.

Case 1.1.2.1: $b_w = b_y$. Then, since S_w hits S_v, $N_{out}(b_y) \subseteq S_w \cup S_v$. Therefore, $H = S_v \cup S_w \cup S_x \cup S_y$ has no out-neighbors to H^c, contradicting that $G = H$ has more than 4 elimination sets.

Case 1.1.2.2: $b_w \neq b_y$. Then, we know that $b_y \notin S_w$ and S_w hits S_y. Since S_w hits S_v, S_y, we see that $S_v \cap S_y \neq \emptyset$. and S_y cannot hit S_v, $b_v \in S_y$. Notice $b_v \in S_x \cap S_y \cap S_v$. Since S_y cannot hit S_x, $b_x \in S_y$. But, $b_x \in S_x \cap S_y \cap S_v^c$ and $w_1 \in S_x \cap S_y^c \cap S_v$ contradicts Lemma 57.

Case 1.1.3: $b_v = b_w$. Then, since $b_v \in S_x$, we see that $b_v \in S_x \cap S_v \cap S_w$. Note that if S_y hits S_w, then $H = S_v \cup S_w \cup S_x \cup S_y$ has no out-neighbors to H^c, contradicting that $G = H$ has more than 4 elimination sets. Therefore, $b_w \in S_y$ and $b_w \in S_w \cap S_x \cap S_y$. Since S_y cannot hit S_x, it also follows that $b_x \in S_y$. But, $b_x \in S_x \cap S_y \cap S_w^c$ and $w_2 \in S_x^c \cap S_y \cap S_w$, contradicting Lemma 57.

Case 1.2: u_n is contained in S_y and u_n is not the bad vertex. Then, $N_{out}(u_n) \subseteq S_y$, but $w_1 \in S_v \cap S_w^c \cap S_y$ and $w_2 \in S_v^c \cap S_w \cap S_y$ contradicts Lemma 57.

We conclude that $u_n \in S_0$ or that it is a bad vertex of a non A-type and they have out-neighbors in $S_v \cap S_w^c$ and $S_v^c \cap S_w$. Note that $N_{out}(u_n) \subseteq S_v \cup S_w$. Thus, either 1) or 2) holds.

Case 2: $v_n \notin S_v \cup S_w$. Note that since $v_n \notin S_v \cup S_w$ and n is minimal, we conclude that $u_n = b_v$ or $u_n = b_w$. We claim that $u_n = b_u$. Assume for contradiction that $u_n \neq b_u$, then u_n must be the bad vertex of an A-type in M_u. First, we realize that $N_{out}(u_n) \subseteq S_v \cup S_w \cup \{v_n\}$. Now, we split into cases.

Case 2.1: $u_n = b_v$. Since $N_{out}(u_n) \subseteq S_v \cup S_w \cup \{v_n\}$, we know that u_n must have an out-neighbor w_1 in $S_w \cap S_v^c$. Since S_v is an A-type, it hits S_x and $v_n \in S_x$. Note that if S_x hits S_w, then $H = S_x \cup S_w \cup S_v$ has no out-neighbors in H^c, contradicting our assumption. Therefore, $b_w \in S_x$.

Also, note that S_v hits S_w is impossible since b_v would have been reduced. So, either $b_v = b_w$ or S_w hits S_v.

Case 2.1.1: $u_n = b_v = b_w$. In this case, S_v, S_w are both A-types that hit S_x. However since

$u_n = b_w$, u_n has at least two out-neighbors in S_w^c. Note that if both are in S_v^c, then b_v cannot be reduced. So, u_n has an out-neighbor $w_2 \in S_w^c \cap S_v$. But, $b_w \in S_x \cap S_v \cap S_w$ and $w_1 \in S_w \cap S_v^c \cap S_x$ and $w_2 \in S_w^c \cap S_v \cap S_x$, contradicting Lemma 57.

Case 2.1.2: S_w hits S_v. Note that S_v cannot hit S_w, so $b_w \in S_v$. Since $w_1 \in S_x \cap S_w$ and S_x cannot hit S_w, we conclude that $b_w \in S_x$. Therefore, $b_w \in S_v \cap S_w \cap S_x$. So, either S_w hits S_x or $b_w = b_x$.

Case 2.1.2.1: S_w hits S_x. Then, S_w hits S_x, S_v. So, S_w must have out-neighbors $w_2 \in S_w^c \cap S_x \cap S_v$. But, $w_1 \in S_w \cap S_x \cap S_v^c$, contradicting Lemma 57.

Case 2.1.2.2: $b_w = b_x$. Then, $N_{out}(b_x) \subseteq S_v \cup S_w$, so $H = S_v \cup S_w \cup S_x$ has no out-neighbors in H^c, contradicting that G has more than 4 elimination sets.

Case 2.2: $u_n = b_w$. Note that the same argument for Case 2.1 holds for Case 2.2.

We conclude that $u_n = b_u$ and b_u has at most one out-neighbor outside $(S_u \cup S_w)^c$. □

Theorem 91. *Let M_u be a minimally connected set with more than 4 elimination sets. Then, M_u does not have 2 out-neighbors in $(M_u \cup T_u)^c$.*

Proof. Assume that M_u has 2 out-neighbors in $(M_u \cup T_u)^c$. By Lemma 83, we have M_u, M_v such that $M_u \cup T_u$ and $M_v \cup T_v$ hit each other. Furthermore, M_v has 2 out-neighbors in $(M_v \cup T_v)^c$.

If M_v has 2 out-neighbors in M_u and first assume they are contained in (possibly non-distinct) elimination sets $S_x, S_y \subseteq M_u$ such that either $b_x = b_y$, or S_x hits S_y or S_y hits S_x. Let us call this the containment property. Now, if the two out-neighbors of M_u in M_u^c are contained in some elimination set S_z, then there exists $a \in M_u$ such that $(u, a) \in E(G)$. Otherwise, a is just an empty vertex. Also, note that $T_u = \emptyset$ since M_u, M_v cannot hit T_u by Lemma 80.

Now, out-contract vertices in $M_u \cap S_w^c \cap S_x^c \cap \{a, b_u\}^c$. If there are 3 elimination sets in M_u, then the previous set was certain non-empty due to the existence of lonely vertices. Now, we appeal to the characterization theorem with $V_0 = S_0$, $V_j = S_{v_j} \cap M_u^c$ for $1 \leq j \leq s$, and $V_{s+1} = S_w \cup \{a, b_u\}$ and $V_{s+2} = S_x \cup \{a, b_u\}$. Checking the properties in the characterization theorem, we see that in G', we still have directed pathwidth≥ 2, a contradiction.

So, the containment property does not hold. Then, assume there exists a $S_w \subseteq M_u$ such that $d(S_w, S_u) \geq 2$. By Lemma 90, we can find S_x such that either S_x hits S_w or S_w hits S_x or $b_w = b_x$. Note that $S_x \subseteq M_u$ or else S_w would contain all the out-neighbors of M_u and the containment property holds. And, we claim that there is a vertex $y \in T_u$ that has out-neighbors in $S_x \cap S_w^c$ and $S_x^c \cap S_w$. By Lemma 90, we have the following cases:

Case 1: There exists a non A-type S_y such that b_y has out-neighbors in $S_x \cap S_w^c$ and $S_x^c \cap S_w$ and $N_{out}(b_y) \subseteq S_x \cup S_w$. It suffices to show that S_y shares a bad vertex with S_v. Since there are only minimally connected sets, we see that S_y must share a bad vertex with S_u or S_v. Assume that the former holds.

Let H be the set of elimination sets that have bad vertex b_u. Since M_u already has 2 out-neighbors in $(M_u \cup T_u)^c$, by Lemma 80, M_u has no out-neighbor in $H^c \cap M_u$. Therefore, if the former does hold, then b_u has an out-neighbor $w_1 \in S_x^c \cap S_w$ and $w_1 \in H$ is forced. But, H only contain sets of distance 0 and by Lemma 75, S_w cannot have distance ≥ 2. Then, M_u has both out-neighbors in S_x, S_w, and so the containment property holds.

Case 2: There exists $y \in S_0$ such that such that y has out-neighbors in $S_x \cap S_w^c$ and $S_x^c \cap S_w$ and $N_{out}(y) \subseteq S_x \cup S_w$. Since $S_x \subseteq M_u$, we conclude that $y \in T_u$.

Case 3: $b_x = b_u$ and b_u has an out-neighbor in $S_w \cap S_x^c$ and b_u has only 1 out-neighbor in $(S_w \cup S_x)^c$. Since b_u has an out-neighbor in S_w, which is impossible by similar argumentation in Case 1.

Case 4: $b_w = b_u$ and b_u has an out-neighbor in $S_x \cap S_w^c$ and b_u has only 1 out-neighbor in $(S_w \cup S_x)^c$. This is not possible because if $b_w = b_u$, then S_w has distance 0, a contradiction.

So, our claim holds and M_v must have an out-neighbor in T_u. If M_v has 1 out-neighbor in M_u, then note that the out-neighbor must be in $S_x \cup S_w$. Otherwise, we can reduce T_u and we would still have a graph of directed pathwidth ≥ 2, by the characterization theorem with $V_0 = S_0 \cap T_u^c$ and $V_j = S_{v_j}$.

However, we can follow the same procedure as described above to contract $M_u \cap S_w^c \cap S_x^c \cap \{a, b_u\}^c$ and derive a contradiction.

If all $S_w \subseteq M_u$ has $d(S_w, S_u) \leq 1$, then let S_x, S_y be the two elimination sets that contain all out-neighbors of M_v and T_u. Then, let S_x hit S_{w_1} or $S_x = S_{w_1}$, where $b_{w_1} = b_u$. And similarly for S_y, define a S_{w_2}. Then out-contract any vertex in $M_u \cap S_x^c \cap S_y^c \cap S_{w_1}^c \cap S_{w_2}^c \cap \{a, b_u\}^c$ and derive a contradiction. □

Theorem 92. *Let M_u be a minimally connected set with more than 4 elimination sets. Then, M_u does not have any out-neighbors in $(M_u \cup T_u)^c$.*

Proof. Assume that M_u has out-neighbors in $(M_u \cup T_u)^c$. By Theorem 91, we know that M_u has at most 1 out-neighbor in $(M_u \cup T_u)^c$. By Lemma 84, we know that there is at most one other minimally connected set in G.

Case 1: There exists M_v, T_v such that $M_u \cup T_u$ and $M_v \cup T_v$ hits each other and $M_v \cup T_v$ also has 1 out-neighbor in $(M_v \cup T_v)^c$. If M_u has at least 2 out-neighbors in M_u^c, then we can follow the argument in Lemma 91 to deduce our result.

Otherwise M_u only has 1 out-neighbor in M_u^c. This means that $T_u = \emptyset$. Even if M_v hits T_u, then we can still contract T_u and it would still have directed pathwidth ≥ 2.

Now, assume there exists a set $S_w \subseteq M_u$ such that $d(S_w, S_u) \geq 1$. Then by Lemma 90, we can find S_x such that S_w hits S_x or S_x hits S_w or $b_x = b_w$. Furthermore, we have the following cases

Case 1.1: If there exists a non A-type S_y such that b_y has out-neighbors $w_1 \in S_w \cap S_x^c$ and $w_2 \in S_x \cap S_w^c$. Note that $b_y \neq b_v$ since it has only 1 out-neighbor in $(M_v \cup T_v)^c$. Therefore, $b_y = b_u$. Let H be all set of all distance 0 elimination sets in M_u. Note that if $w_1, w_2 \notin H$, then Lemma 80 is contradicted. So, one of $w_1, w_2 \in H$ and we note that $d(S_w, S_u) \leq 2$ or else S_w, S_x would have distance ≥ 2, contradicting Lemma 75.

If exactly one of $w_1, w_2 \notin H$ then WLOG, let it be w_1. If $d(S_w, S_u) = 2$, then we can find S_y, S_z such that S_w hits S_y, which hits S_z, where $b_z = b_u$. We can contract all else in $M_u \cap S_w^c \cap S_y^c \cap S_z^c$ and show by the characterization theorem that directed pathwidth ≥ 2. So, there are at most 3 elimination sets in M_u. We use a similar argument if $d(S_w, S_u) = 1$.

Otherwise, we have $w_1, w_2 \in H$, there exists S_z of distance 0 such that $w_1 \in S_z$. By the same reasoning, we can find S_a of distance 0 such that $w_2 \in S_a$. By Lemma 89, $w_1 \in S_z \cap S_a^c$ and $w_2 \in S_a \cap S_z^c$. Now, we may contract all else in $M_u \cap S_z^c \cap S_a^c$ and show by equivalence that directed pathwidth ≥ 2. So, there are at most two elimination sets in M_u.

Case 1.2: There exists $y \in S_0$ with out-neighbors in $S_w \cup S_x$. However, note that $S_x \subseteq M_u$ since M_v only has 1 out-neighbors in $(M_v \cup T_v)^c$. So, $y \in T_u$ but $T_u = \emptyset$, contradiction.

Case 1.3: $b_w = b_u$ and and b_u has an out-neighbor in $S_x \cap S_w^c$. This contradicts the distance assumption of S_w.

Case 1.4: $b_x = b_u$ and and b_u has 1 out-neighbor in $(S_w \cup S_x)^c$. has an out-neighbor in $w_1 \in S_w \cap S_x^c$. If $w_1 \notin H$, then follow the argument in Case 1.1 to deduce that there are at most 2 elimination sets. Otherwise, $w_1 \in H$. So, we can find S_y that contains w_1 such that $b_y = b_u$. Note that b_u contains two out-neighbors in S_y^c and it already contains an out-neighbor in $M_v \cup T_v$. But, b_u has only 1 out-neighbor in $(S_w \cup S_x)^c$, so b_u has an out-neighbor $w_2 \in S_x \cap S_y^c$. We conclude as in Case 1.1

If we cannot find a S_w with distance ≥ 1, we see that all elimination sets are of distance 0. Since M_u only has 1 out-neighbor in M_u^c, there must exist S_x of distance 0 such that b_u has an out-neighbor in S_x. However, since b_u must have at least 2 out-neighbors in S_x^c, we must be able to find S_y of distance 0 such that b_u has an out-neighbor in S_y. By Lemma 89, these out-neighbors are in $S_x \cap S_y^c$ and $S_x^c \cap S_y$. Then, we contract all else in $M_u \cap S_x^c \cap S_y^c$ and deduce that there are at most two elimination sets in M_u.

Case 2: There does not exist M_v, T_v. Therefore, $(M_u \cup T_u)^c \subseteq S_0$. However, if $v \in S_0 \cap T_u^c$, then v has at most 1 out-neighbor in $M_u \cup T_u$ since if v has both out-neighbors in $M_u \cup T_u$, then by 85, $v \in T_u$ must follow. Thus, we may conclude by similar argument in Case 1. □

Theorem 93. *M_u has at most 4 elimination sets.*

Proof. Assume that M_u has more than 4 elimination sets. Then, by Theorem 91, M_u has no out-neighbors outside $M_u \cup T_u$. So, $G = M_u \cup T_u$.

Let H denote the set of all elimination sets of distance 0 in M_u. If M_u has out-neighbors in M_u^c, then by Lemma 87, there exists only one $v \in T_u$ such that v has out-neighbors in M_u. Let S_x, S_y be the elimination sets that contain the 2 out-neighbors of v. If one of S_x, S_y has distance ≥ 2, then if M_u has an out-neighbor $w \in M_u \cap H^c$, then $w \in S_x \cup S_y$, or else we may just out-contract all vertices in T_u into the elimination set of distance ≥ 2. So, we can follow the cases in Lemma 91.

So, we may assume that M_u has no out-neighbors in M_u^c and $T_u = \emptyset$. By Lemma 80, H has at most 2 out-neighbors in H^c. First note that for any S_v of distance ≥ 2 in M_u, by Lemma 90, there must exists S_w such that b_u has out-neighbors in $S_w \cap S_v^c$ and $S_v \cap S_w^c$.

We claim that M_u has no elimination sets of distance ≥ 4. Assume otherwise and note that we can find $S_v \subseteq M_u$ be an elimination set in M_u such that S_v is not hit by another elimination set and its distance is ≥ 4. Then, by Lemma 90, there must exist S_w such that either S_v hits S_w or $b_w = b_v$ and b_u has out-neighbors in $w_1 \in S_w \cap S_v^c$ and $w_2 \in S_w^c \cap S_v$ and $N_{out}(b_u) \subseteq S_v \cup S_w$. Note that S_v has distance ≥ 3. Note that S_w hits another elimination set S_x, which has distance ≥ 2, so there exists S_y such that b_u has out-neighbors in $w_3 \in S_y^c \cap S_x$ and $w_4 \in S_x \cap S_y$. Note that $w_3 \neq w_4$ and $w_1 \neq w_2$. Since b_u has at most 2 out-neighbors in H^c and $w_1, w_2, w_3 \in H^c$ by Lemma 75, we must have $w_2 = w_3$ and $w_4 \in H$. But $w_4 \in H$ contradicts $N_{out}(b_u) \subseteq S_v \cup S_w$.

If $S_v \subseteq M_u$ is an elimination set of distance ≥ 2, then the same argument above shows that there must exists S_w such that $N_{out}(b_u) \subseteq S_v \cup S_w$. If S_w hits S_v, then since S_v is of distance 2, there exists S_{u_1}, S_{u_2} such that S_v hits S_{u_1} and S_{u_1} hits S_{u_2} with $b_{u_2} = b_u$. By strong connectivity, note that $S_w, S_v, S_{u_1}, S_{u_2}$ contains all its out-neighbors. So, we conclude that there are at most 4 elimination sets.

We can similarly conclude if S_v hits S_w or if $b_w = b_v$. So, we may assume that all elimination sets are of distance ≤ 1. If there exists $S_v \subseteq M_u$ a elimination set of distance 1, then by 90, we have S_w with the given properties and the following cases:

Case 1: Note that b_u has out-neighbors in $S_v \cap S_w^c$ and $S_v^c \cap S_w$ and $N_{out}(b_u) \subseteq S_v \cup S_w$. Let S_v hit S_{u_1} and $b_{u_1} = b_u$, then clearly, S_v, S_w, S_{u_1} contains all its out-neighbors and we conclude there are at most 4 elimination sets

Case 2: This is impossible as $T_u = \emptyset$

Case 3: $b_v = b_w$ is impossible as the distance of S_v is 1.

Case 4: $b_w = b_u$ and b_u has an out-neighbor in $S_v \cap S_w^c$ and b_u has 1 out-neighbor in $w_1 \in (S_w \cup S_v)^c$. Let S_{u_1} contain w_1 and since it is of distance ≤ 1, there exists S_{u_2}, where $S_{u_1} = S_{u_2}$ or S_{u_1} hits S_{u_2} and $b_{u_2} = b_u$. Then, we conclude since $S_v, S_w, S_{u_1}, S_{u_2}$ contains all of its out-neighbors.

So we may assume that $G = H$. Assume we can find S_v, S_w such that there exists out-neighbors of b_u, u_1, u_2, such that $u_1 \in S_v^c \cap S_w$ and $u_2 \in S_{v_1} \cap S_{v_2}^c$. If $G = S_{v_1} \cup S_{v_2}$, then it at most 4 elimination sets. Otherwise, note that there must be an out-neighbor of b_u in $(S_v \cup S_w)^c$ and say $u_3 \in S_x$. Note that if both $u_1, u_2 \in S_x$, then since $b_u \in S_v \cup S_w \cup S_x$, we contradict Lemma 57. Without loss of generality, let $u_1 \in S_v^c \cap S_w \cap S_x^c$. By Lemma 88, we have at most 3 elimination sets since $u_3 \in S_v^c \cap S_w^c \cap S_x$.

If we cannot find such S_v, S_w, then we claim that we can find one elimination set S_x that contain $N_{out}(b_u)$. Assume otherwise, then we can find S_y such that $|S_y \cap N_{out}(b_u)|$ is maximized. By our assumption, we can find $u_1 \in N_{out}(b_u)$ such that $u_1 \notin S_y$. However, we can also find S_z such that $u_1 \in S_z$. Note that if there exists $u_i \in S_y \cap S_z^c$, then we are done. So, we conclude that $|S_y \cap N_{out}(b_u)| < |S_z \cap N_{out}(b_u)|$, contradicting our assumption. Therefore, S_x contains all the out-neighbors of b_u and we have only 1 elimination set in G. □

8.3 Finiteness of Forbidden Minors

Theorem 94. *Let S_u be an elimination set, then $|S_u| \leq 160000$.*

Proof. We know that there exists a simple directed path from u to b_u by following the elimination ordering (and disregarding in-contractions), let it be P_{ub_u}. We call this the main path. We first claim that this path is bounded. Let $v_1, ..., v_k$ be vertices on this path, where $v_1 = u, v_k = b_u$. Then, for $1 \leq i < k$, consider the graph obtained by out-contracting (v_i, v_{i+1}), say G'. Note that if G' is not strongly connected, then we can consider strongly connected components with as long as all bad vertices are still in the strongly connected component.

We want to appeal to the characterization theorem with $V_0 = S_0, V_j = S_{w_j} \cap G'$ for $1 \leq j \leq s$. The first few properties hold trivially. Note that S_0 has vertices of outdegree ≥ 2 since if $x \in S_0$ has lost outdegree, then $N_{out}(x) \subseteq S_u$, which would imply $x \in S_v$, a contradiction. So, by the characterization theorem, there must exists $V_j = S_{v_j}$ such that either property 3 fails or property 4 fails.

If property 3 fails, then we see that either b_{w_j} had $v_i, v_{i+1} \in V_j^c$ as out-neighbors, $b_{w_j} = v_i$ or $b_{w_j} = v_{i+1}$ and it has $v_i \in V_j^c$ as an out-neighbor. If property 4 fails, then it must be that $v_i \in V_j^c, v_{i+1} \in V_j$. But, note that the number of elimination sets and bad vertices are bounded to 8. Note that by if $v_i \in V_j^c, v_{i+1} \in V_j$, then $v_k \in V_j$ unless there exists l such that $v_l = b_{w_j}$. Therefore, we conclude that for each bad vertex b_{w_j} can only prevent the contraction of at most 16 vertices. Furthermore, the second condition can only prevent the contraction of at most 8 vertices. We also have to make sure that we do not disconnect an elimination set from its bad vertex. It suffices to show that for any elimination set S_x, P_{xb_x} is still a directed path. This is only a problem when v_i is an intersection vertex for P_{xb_x} and P_{ub_u} and v_{i+1} is not, since if both v_i, v_{i+1} are intersection points, then we may either cycle contract or out-contract without problems. However, consider contracting v_{i+1} and note that we run into a problem only if property 3 fails and v_{i+1}, v_{i+2} is a bad vertex, so there are at most 8 more possible vertices when this can be a problem. Since there are 8 different paths P_{xb_x}, gives an additional 8*8 = 64 vertices. Therefore, P_{ub_u} has at most 100 vertices.

Consider the maximal elimination ordering of S_u, say $u_1, ..., u_m$ and it outsequence $w_1, ..., w_m$. If there exists a vertex x such that $w_i, ..., w_{i+k} = x$, then we claim that $k \leq 25$. Now, consider $u_{i+1}, ..., u_{i+k}$ and find a u_{i+l} such that it does not have $u_{i+1}, ..., u_{i+k}$ as its in-neighbor. Consider contracting directed path from u_{i+l} to x in G. Note that property 3 holds, we can find $V_j = S_{v_j}$ such that unless a bad vertex is on that path or that the bad vertex is w and one of its two out-neighbors in V_j^c is on that path. Note that property 4 holds since u_{i+l} does not have $u_{i+1}, ..., u_{i+k}$ as its in-neighbor. So, there are at most 16 such paths. By the same argument before, there are at most 100 vertices on these paths.

So, our most conservative bound is that for each 100 vertices on P_{ub_u}, there could be these 16 paths that are reduced at each vertex along P_{ub_u}, each of which has 100 vertices. Hence S_u has at most 100*16*100 = 160000 vertices. □

Theorem 95. *Let G be a forbidden minor, then $|S_0| \leq 7$*

Proof. Note that if there exists two minimally connected sets, M_u, M_v, then by the argument in Lemma 83, Lemma 84, we see that $|S_0 \cap T_u^c \cap T_v^c| \leq 1$. Since $|T_u|, |T_v| \leq 3$ by Lemma 87, we conclude that $|S_0| \leq 7$.

Otherwise, note that if no minimally connected sets exist, then we must have A-types that hit each other in a cycle. This implies that $S_0 = \emptyset$. This leaves us the case when we only have one minimally connected set M_u. Note that if $(M_u \cup T_u)^c = \emptyset$, then $S_0 = T_u$, so $|S_0| \leq 3$.

Otherwise, $M_u \cup T_u$ only has 1 out-neighbor $x \in (M_u \cup T_u)^c = S_0 \cap T_u^c$. Let us consider the case when $(b_u, x) \notin E(G)$ and $(b_u, x) \in E(G)$. Now, by Lemma 86, note that if we can find $u, v \in S_0 \cap T_u^c$, such that out-contracting (u, v) still preserves the outdegree of the vertices in $S_0' \cap T_u^c$, then we derive a contradiction. Since $|T_u| \leq 3$, we conclude that $|S_0| \leq 7$. □

Theorem 96. *$\mathcal{F}'$ is finite.*

Proof. It suffices to show that $\mathcal{F}'$ only contains graphs with a bounded vertex count. Let $G \subseteq \mathcal{F}'$. By Lemma 84 and Theorem 93, we conclude that there are at most 8 elimination sets in G. So, by Theorem 94 and Theorem 95, we conclude that $|V(G)| \leq |S_0| + \sum_{i=1}^{s} |S_{v_i}|$ is bounded. □

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